Exercise 1. We want to find all rational numbers $x, y \in \mathbb{Q}$ such that $x^2+y^2=1$, i.e., all $\alpha = x+iy \in \mathbb{Q}(i)$ s.t. $\alpha \in \ker(N_{L/\mathbb{Q}} : \mathbb{Q}(i)^* \to \mathbb{Q}^*)$.

By Hilbert's 90, $N(\alpha)=1 \iff \alpha = \frac{\sigma\beta}{\beta} = \frac{x-iy}{x+iy}$

where $\sigma \in G$ is complex conjugation.

So the rational points on the unit circle are of the form $\left(\frac{x^2-y^2}{x^2+y^2}, \frac{2xy}{x^2+y^2}\right)$

for $x, y \in \mathbb{Q}$.

Exercise 2 See Prop. 1.30 - Cor. 1.32 in Milne's notes.

Exercise 4. Since $\mu_n \subseteq K$, G acts trivially on μ_n. Hence a crossed homomorphism $G \to \mu_n$ is the same as an ordinary homomorphism and we have $H^1(G, \mu_n) = \mathrm{Hom}(G, \mu_n)$.

2)

To compute $H^1(G, \mu_n)$ we the exact sequence & Hilbert 90 to get the long exact cohomology sequence

$$\begin{tikzcd}
1 \arrow[r] & H^0(G,\mu_n) \arrow[r] \arrow[d, equal] & H^0(G,L^*) \arrow[r, "n"] \arrow[d, equal] & H^0(G,L^{*n}) \arrow[r] \arrow[d, equal] & H^1(G,\mu_n) \arrow[r] & H^1(G,L^*) \arrow[d, equal] \\
1 \arrow[r] & \mu_n \arrow[r] & (L^*)^G \arrow[r, "(-)^n"'] & (L^{*n})^G \arrow[r] & H^1(G,\mu_n) \arrow[r] & 1
\end{tikzcd}$$

Hence $H^1(G,\mu_n) \cong \operatorname{coker}\left((-)^n : (L^*)^G \to (L^{*n})^G\right)$

$$= \frac{K^* \cap L^{*n}}{K^{*n}}.$$

Exercise 5. See Theorem 5.30 in Milne's "Field Theory" notes