Exercise 1

By Hilbert's 90, $H^1_T(G, \ell^*) = 0$.

Since ℓ^* is finite, $h(\ell^*) = 1$,

hence $H^0_T(G, \ell^*)$.

Since G is cyclic, we conclude by periodicity.

Exercise 2

Let ℓ/k be an extension of finite fields, say $|k| = q$ and $|\ell| = q^f$.

Now ℓ^* is cyclic of order $q^f - 1$, say generated by $\alpha \in \ell$. Then

$$Nm_{\ell/k}(\alpha) = \alpha^{1+q+q^2+\cdots+q^{f-1}} \in k^*.$$

Claim: $Nm_{\ell/k}(\alpha)$ generates k^*.

Indeed, $Nm_{\ell/k}(\alpha)$ has order $q-1 = |k^*|$,

Since $\alpha^{(q-1)(1+q+\cdots+q^{f-1})} = \alpha^{q^f-1} = 1$, and the order is $\geq q-1$.

Hence $Nm_{\ell/k}$ is surjective.

2)

Exercise 3

a)
$G = Gal(\mathbb{C}/\mathbb{R})$ is cyclic, so $H^r_T(G, \mathbb{C}^*)$ will be 2-periodic.

By Hilbert's 90, $H^1_T(G, \mathbb{C}^*) = 0$.

On the other hand,

$$H^0_T(G, \mathbb{C}^*) = \frac{\mathbb{R}^*}{Nm(\mathbb{C}^*)} = \frac{\mathbb{R}^*}{\mathbb{R}^*_{>0}} \cong \mathbb{Z}/2.$$

Hence

$$H^r_T(G, \mathbb{C}^*) \cong \begin{cases} \mathbb{Z}/2, & r \equiv 0 \ (2) \\ 0, & r \equiv 1 \ (2). \end{cases}$$

b) The isomorphism

$$inv_{\mathbb{C}/\mathbb{R}} : H^2_T(G, \mathbb{C}^*) \cong \mathbb{Z}/2 \xrightarrow{\cong} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

sends the generator $1 \in \mathbb{Z}/2$ to the generator $\frac{1}{2} \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, and 0 to 0.

(The generator $1 \in \mathbb{Z}/2$ corresponds to the class of -1 in $\mathbb{R}^*/Nm(\mathbb{C}^*)$).

Exercise 4

Let $x \otimes y \in M \otimes N$ and suppose x is a norm, say $x = \sum_{\sigma \in G} \sigma m$, $m \in M$

Then
$$x \otimes y = \left(\sum_{\sigma} \sigma m\right) \otimes y$$
$$= \sum_{\sigma} \sigma m \otimes y$$
$$= \sum_{\sigma} \sigma m \otimes \sigma y$$
$$= Nm_G(m \otimes y).$$

Similarly if y is a norm.