Exercise 1 If L/K is ramified, the norm $Nm: U_L \to U_K$ will not be surjective, hence $H_T^0(G, U_L) \neq 0$.

For instance, let p be an odd prime and consider $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\sqrt{-p})$.

Then L/K is a ramified quadratic extension, and we have seen that

$$Nm(L^*) = \langle p \rangle \oplus \langle \zeta_{p-1}^2 \rangle \oplus U_p^1$$

Hence $\zeta_{p-1} \notin Nm(U_L)$, so $Nm(U_L) \neq U_K$,

and thus $H_T^0(G, U_L) = \dfrac{U_K}{Nm(U_L)} \neq 0$.

Exercise 2 b) We have that $Nm_{L/K_v}(1 - \xi \sqrt[d]{a}) = 1 - \xi^d a$,

hence

$$Nm\left(\prod_{i=1}^{n/d} (1 - \zeta_n^i \sqrt[d]{a})\right) = \prod_{i=1}^{n/d} Nm(1 - \zeta_n^i \sqrt[d]{a})$$

$$= \prod_{i=1}^{n/d} (1 - \zeta_n^{di} a)$$

$$= 1 - a^{n/d}$$

$$= 1 - x.$$

2) c)

(i) $(1,x) = (1\cdot 1, x) = (1,x)(1,x)$

$\Rightarrow (1,x) = 1$, and similarly for $(x,1)$.

(ii) $1 = \left(\frac{1}{x}, 1-\frac{1}{x}\right) = \left(x, 1-\frac{1}{x}\right)^{-1}$

$= \left(x, \frac{1-x}{-x}\right)^{-1}$

$= (x, 1-x)^{-1}(x,-x)$

$= (x,-x)$

using $(x,y)(x^{-1},y) = (1,y) = 1$

$\Rightarrow (x^{-1},y) = (x,y)^{-1}$

(iii) $1 = (xy, -xy)$

$= (x,-x)(x,y)(y,x)(y,-y)$

$= (x,y)(y,x)$

(iv) $1 = (x,-x) = (x,x)(x,-1)$.

d) For $x,y \in \mathbb{R}^*$, $(x,y)_\infty = \frac{\sigma(\sqrt{x})}{\sqrt{x}}$ where

$\sigma = (-, \mathbb{R}(\sqrt{y})/\mathbb{R}) = \begin{cases} \mathrm{id} & \text{if } y>0 \text{ (since then } \mathbb{R}(\sqrt{y}) = \mathbb{R}) \\ c & \text{if } y<0, \text{ where } c = \text{generator of } \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \end{cases}$

(Since $\mathbb{R}(\sqrt{y}) = \mathbb{C}$ if $y<0$).

Hence $(x,y)_\infty = 1$ if $y>0$, while if $y<0$, we have

$(x,y)_\infty = \frac{c(\sqrt{x})}{\sqrt{x}} = \begin{cases} \frac{\sqrt{x}}{\sqrt{x}} = 1 & \text{if } x>0 \\ \frac{-i\sqrt{|x|}}{i\sqrt{|x|}} = -1 & \text{if } x<0. \end{cases}$

All in all, $(x,y)_\infty = \begin{cases} 1 & \text{if } x>0 \text{ or } y>0 \\ -1 & \text{if } x<0 \text{ and } y<0 \end{cases}$

Exercise 3

a) If $F = \mathbb{F}_q$ is a finite field,

then $F^\times$ is cyclic, say $F^\times = \langle \zeta \rangle$.

Since $\{\zeta^n, \zeta^m\} = \{\zeta, \zeta\}^{nm} \in K_2(F)$,
it suffices to show $\{\zeta, \zeta\} = 1$.

Now, the equation $\zeta X^2 = 1 - \zeta Y^2$

has a solution $(x, y) \in F^* \times F^*$, since

$|\{\zeta x^2 \mid x \in F\}| + |\{1 - \zeta y^2 \mid y \in F\}| > q$.

(We must have $x, y \in F^*$ since ζ is not a square).

Then $1 = \{\zeta x^2, 1 - \zeta x^2\} = \{\zeta x^2, \zeta y^2\}$

$$= \{\zeta, \zeta\}\{\zeta, y\}^2\{x, \zeta\}^2\{x, y\}^4,$$

and this equals $\{\zeta, \zeta\}$ because the square $\{a, b\}^2$

of any element $\{a, b\} \in K_2(F)$ is trivial: indeed,

if $a = \zeta^n$, $b = \zeta^m$, then

$\{a, b\}^2 = \{\zeta, \zeta\}^{2nm} = 1$ since $\{\zeta, \zeta\} = \{\zeta, \zeta\}^{-1} \Rightarrow \{\zeta, \zeta\}^2 = 1$

(If F has characteristic 2 then $\zeta = -\zeta$, so
$\{\zeta, \zeta\} = \{\zeta, -\zeta\} = 1$).

4)

b) For brevity let's write just $v(x)$ for $\mathrm{ord}_v(x)$.

We see that ∂_v is bimultiplicative and that $\partial_v(x,y) \in k(v)^*$.

We show that it satisfies the Steinberg relation.

Case 1: $v(x) > 0$, then $x \in \mathfrak{p}_v$, and $1-x \equiv 1 \mod \mathfrak{p}_v$,

so $v(1-x) = 0$. Then

$$\partial_v(x, 1-x) = (-1)^{v(x)v(1-x)} \frac{x^{v(1-x)}}{(1-x)^{v(x)}} \mod \mathfrak{p}_v$$

$$= \frac{1}{(1-x)^{v(x)}} \equiv 1 \mod \mathfrak{p}_v.$$

Similarly if $v(1-x) > 0$.

Case 2: $v(x) < 0$. Then $\frac{1}{x} \in \mathfrak{p}_v$,

and $\frac{1-x}{x} = -1 + \frac{1}{x} \equiv -1 \mod \mathfrak{p}_v$.

Hence $v\left(\frac{1-x}{x}\right) = 0 = v(1-x) - v(x)$,

So $v(1-x) = v(x)$.

Then we get

$$\partial_v(x, 1-x) = (-1)^{v(x)v(1-x)} \frac{x^{v(1-x)}}{(1-x)^{v(x)}} \mod \mathfrak{p}_v$$

$$= (-1)^{v(x)^2} \left(\frac{x}{1-x}\right)^{v(x)} \equiv (-1)^{v(x)} (-1)^{v(x)} \equiv 1 \mod \mathfrak{p}_v.$$

Similarly if $v(1-x) < 0$.

Case 3: $v(x) = v(1-x) = 0$, immediate.

c) Since $\text{ord}_p(-1) = 0 \ \forall p$,

$\{-1,-1\} \in \ker \partial$. To show that

$\{-1,-1\} \neq 1$ in $K_2(\mathbb{Q})$, we must find

a symbol on $\mathbb{Q}$ which is nontrivial on $\{-1,-1\}$.

Consider the Hilbert symbol $(-,-)_\infty$ at ∞,

then $(-1,-1)_\infty = -1$ by 2 d). Hence $\{-1,-1\} \neq 1$

in $K_2(\mathbb{Q})$.

In fact ∂ is surjective with kernel generated

by $\{-1,-1\}$; since $\{-1,-1\}$ has order 2 in $K_2(\mathbb{Q})$

the kernel is isomorphic to μ_2, and

$$K_2(\mathbb{Q}) \cong \mu_2 \oplus \bigoplus_{p \text{ prime}} \mathbb{F}_p^*.$$

We will prove this later in the course,

and see that this result is intimately

linked with quadratic reciprocity.

6) Exercise 4

a) We know that $\mathbb{Q}_p^* \cong p^{\mathbb{Z}} \oplus U_p$, and

$\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p$ is totally ramified of degree

$p^{n-1}(p-1) = \varphi(p^n) = |(\mathbb{Z}/p^n)^*|$.

Let ζ_n be a primitive n-th root of unity,

then $Nm(1-\zeta_n) = \Phi_{p^n}(1) = p$, hence

$p \in N = Nm(\mathbb{Q}_p(\mu_{p^n})^*)$, and thus $p^{\mathbb{Z}} \subseteq N$.

On the other hand, $U_p/U_p^n \cong (\mathbb{Z}/p^n)^*$,

Since the map $\begin{array}{ccc} U_p & \to & (\mathbb{Z}/p^n)^* \\ u & \mapsto & u \bmod p^n \end{array}$ has kernel U_p^n.

Hence $p^{\mathbb{Z}} \oplus U_p^n$ has index $|(\mathbb{Z}/p^n)^*| = p^{n-1}(p-1)$

in $\mathbb{Q}_p^*$, so $N = p^{\mathbb{Z}} \oplus U_p^n$.

b) Let $L/\mathbb{Q}_p$ be a finite abelian extension.

For big enough f, n we have

$$\langle p^f \rangle \oplus U_p^n \subseteq Nm_{L/\mathbb{Q}_p}(L^*)$$

Now $\langle p^f \rangle \oplus U_p^n$ has finite index in $\mathbb{Q}_p^*$,

hence corresponds to an extension $E/\mathbb{Q}_p$.

Since the local class field theory correspondence is inclusion reversing we have $L \subseteq E$.

Claim: E is cyclotomic.

Indeed, $\langle p^f \rangle \oplus U_p^n = (\langle p^f \rangle \oplus U_p) \cap (\langle p \rangle \oplus U_p^n)$,

So $E = E_1 \cdot E_2$ where $E_1 \longleftrightarrow \langle p^f \rangle \oplus U_p$,

$E_2 \longleftrightarrow \langle p \rangle \oplus U_p^n$.

E_1 is the unique unramified extension of $\mathbb{Q}_p$ of degree f (since its norm group contains U_p)

hence $E_1 = \mathbb{Q}_p(\zeta_{p^f-1})$. By (a), $E_2 = \mathbb{Q}_p(\zeta_{p^f})$.

Hence $E = \mathbb{Q}_p\left(\zeta_{(p^f-1)p^n}\right)$.

8)

Exercise 5

a) We have $M_w = L_v(\mu_n)$

$$= \mathbb{Q}_p(\mu_{p^{e_p} \cdot m}) = \mathbb{Q}_p(\mu_{p^{e_p}})\mathbb{Q}_p(\mu_m)$$

b)

$\mathbb{Q}_p(\mu_m)$ is the maximal unramified subextension of $M_w/\mathbb{Q}_p$, hence

$$I_p \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^{e_p}})/\mathbb{Q}_p)$$

c) The fixed field of I is an unramified extension of $\mathbb{Q}$, hence equals $\mathbb{Q}$. Therefore $I = \mathrm{Gal}(M/\mathbb{Q})$.

d) We have that

$$|I| \leq \prod_{p \in S} |I_p| = \prod_p \varphi(p^{e_p})$$

$$= \varphi(n) = [\mathbb{Q}(\mu_n) : \mathbb{Q}].$$

Hence $[M : \mathbb{Q}] = [\mathbb{Q}(\mu_n) : \mathbb{Q}]$, hence $M = \mathbb{Q}(\mu_n)$.

By definition $L \subseteq M = L(\mu_n)$, so $L \subseteq \mathbb{Q}(\mu_n)$.