MAT4250 STARTING QUESTIONS

Choose one of the four problems below to present in the beginning of the oral exam (all problems carry equal weight). Your presentation should take about 10 minutes.

You are quite free to include or exclude as much material as you feel necessary to answer the question within the given time frame. Check beforehand that your presentation takes about 10 minutes.

Problem 1. For L/K an abelian extension of number fields, let v denote a finite place of K and w a place of L above v.

(a) Formulate the local reciprocity law for the extension L_w/K_v. Define the local conductor $f(L_w/K_v)$.

(b) Let $\mathfrak{f}(L/K)$ be the global conductor, and denote by $\mathfrak{m}_\infty$ the product of the ramified real places of K. Using that

$$\mathfrak{f}(L/K) = \mathfrak{m}_\infty \prod_v \mathfrak{p}_v^{f(L_w/K_v)}$$

(which you need not prove), explain that $\mathfrak{f}(L/K)$ is divisible precisely by the ramified places.

Problem 2.

(a) Formulate the global reciprocity law in terms of idèles. What are the advantages of the idèlic formulation?

(b) What is $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$? Can you identify $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ with a quotient of $\mathbb{I}_\mathbb{Q}$?

Problem 3.

(a) Talk about Hilbert class fields. Mention Furtwängler's theorem and the principal ideal theorem.

(b) Let $K = \mathbb{Q}(\sqrt{-5})$ and $\mathfrak{p} = (2, 1 + \sqrt{-5}) \subseteq \mathcal{O}_K$. Show explicitly that $\mathfrak{p}$ becomes principal in $K' = \mathbb{Q}(\sqrt{-5}, \sqrt{-1})$ by showing $\mathfrak{p}\mathcal{O}_{K'} = (1 + \sqrt{-1})\mathcal{O}_{K'}$. (You do not need to prove that K' is the Hilbert class field of K.)

Problem 4.

(a) Talk about Dirichlet L-series and Dirichlet's theorem on densities of primes in arithmetic progressions. Mention also Chebotarev's theorem.

(b) Use Dirichlet's or Chebotarev's density theorem to show that the set of primes splitting in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ has density $1/\varphi(m)$.