Ark5: Exercises for MAT4270 — Lie algebras, exponential map and invariant integrals.

THIS SHEET CONCERNS THE WEEK SEPTEMBER 24 TO SEPTEMBER 28

Plans are approximatly the following:

On Tuesday Sep 25 I did: Characters and trace, basic formulas for characters, orthogonal relations, started with finite groups

On Friday Sep 28 I plan to do: More on finite groups. Examples, *i.e.,* symmetric groups.

Exercises

PROBLEM 1. Show the product rule:

$$d_{a(t)b(t)} d_t ab = d_{b(t)} \lambda_{a(t)} d_t b + d_{a(t)} \rho_{b(t)} d_t a$$

where $a\colon \mathbb{R} \to G$ and $b\colon \mathbb{R} \to G$ are two maps from $\mathbb{R}$ into the Lie group G and $ab\colon \mathbb{R} \to G$ is the map $t \mapsto a(t) \cdot b(t)$.

PROBLEM 2.

a) Assume G to be connected. Show that the image $\exp \operatorname{Lie} G$ is *not* a subgroup unless exp is surjective.

b) Show that the product of two 2×2 matrices in $\mathrm{Sl}(2, \mathbb{R})$ with positive eigenvalues has positive eigenvalues.

c) Give an example of matrices a and b with purely imaginary eigenvalues such that the product has negative real eigenvalues.

d) Recall the elements a and b from $\mathrm{Sl}(2, \mathbb{R})$ as in the example in Notes5:

$$a = \begin{pmatrix} -x & y \\ -y^{-1} & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}. \end{pmatrix}$$

Convince yourself that ba is not in the image of the exponetial.

e) Very open question: Find v and w in $\mathrm{sl}(2, \mathbb{R})$ such that $\exp v = a$ and $\exp w = b$. What can you say about the *right side* in the BCH-formula evaluated at that v and w?

Problem 3. *(The centraliser).*Let G be a Lie group and let $g \in G$ be an element. Recall the conjugation map $c_g\colon G \to G$ with $c_g(x) = gxg^{-1}$. Let $\chi_g(x) = c_g(x)x^{-1} = gxg^{-1}x^{-1}$.

a) Show that for any $h \in G$ there is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\chi_g} & G \\ \Big\downarrow{\scriptstyle \lambda_h} & & \Big\downarrow{\scriptstyle \lambda_{ghg^{-1}} \circ \rho_{h^{-1}}} \\ G & \xrightarrow[\chi_g]{} & G \end{array}$$

and use this to show that the rank of χ_g is the same everywhere in G.

b) Show that $\chi_g^{-1}(e)$ is equal to the centraliser $C_G(g)$ and that the centraliser is a closed subgroup. Show that $\operatorname{Lie}_e C_G(g) = \{\, v \in \operatorname{Lie} G \mid \operatorname{Ad}_g v = v \,\}$.

Problem 4. *(Fixed points).*Let $\phi\colon G \to G$ be an automorphism of the Lie group G. Show that the set of fixed points $F(\phi)$ of ϕ, that is the points for which $\phi(x) = x$, is a closed subgroup of G, and that its Lie algebra is given as $\operatorname{Lie} F(\phi) = \{\, v \in \operatorname{Lie} G \mid d_e\phi(v) = v \,\}$.

Problem 5. *(Conjugacy classes of* $\operatorname{Sl}(2, \mathbb{K})$*).*Assume that the two elements a and b of $\operatorname{Sl}(2, \mathbb{K})$ have the same trace τ; where as usual $\mathbb{K}$ is one of the fields $\mathbb{C}$ or $\mathbb{R}$.

a) Show that if $\tau \neq \pm 2$, then a and b are conjugate. Find representatives in each class.

b) Show that if $\tau = 2$ or $\tau = -2$ there are exactly two conjugacy classes having trace τ in each case. Find representives on each class.

Problem 6.

a) Show that any element in SO(3) is conjugate to one of the form

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

b) What are the conjugacy classes of SO(3)? And what are the conjugacy classes of O(3)?

PROBLEM 7. Let H_3 be the Heisenberg group, *i.e.,* the subgroup of upper triangular 3×3-matrices with ones along the diagonal. Show that $\exp\colon \operatorname{Lie} H_3 \to H_3$ is a diffeomorphism.

PROBLEM 8. Let h, x_+, x_- be the usual basis for $\mathrm{Sl}(2, \mathbb{R})$; that is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad x_x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad x_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

a) Compute the matrices of ad_h, ad_{x_0} and ad_{x_1} in that basis.

Let a_t be the element in $\mathrm{Sl}(2, \mathbb{R})$ given by

$$a_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

b) Show that $N = \{ a_t \mid t \in \mathbb{R} \}$ is a Lie subgroup in the strong sense of $\mathrm{Sl}(2, \mathbb{R})$.

c) Determine the Lie subalgebra $\operatorname{Lie} N$ of $\mathrm{sl}(2, \mathbb{R})$.

d) Show that the matrix of $\operatorname{Ad} a_t$ in the usual basis h, x_+, x_- for $\mathrm{sl}(2, \mathbb{R})$ is

$$\begin{pmatrix} 1 & o & t \\ -2t & 1 & -t^2 \\ 0 & 0 & 1 \end{pmatrix}$$

PROBLEM 9. let dX be the usual Lebesque-measure on the space of $n \times n$-matrices $\mathrm{M_n}(\mathbb{R})$ (which is equal to $\mathbb{R}^{n^2}$). Show that $|\det X|^{-n}\, dX$ is both left and right invariant measur on $\mathrm{Gl}(n, \mathbb{R})$.

PROBLEM 10. Let G be the Lie subgroup of $\mathrm{Gl}(2, \mathbb{R})$ whose elements are the matrices of the form

$$\begin{pmatrix} s & u \\ 0 & 1 \end{pmatrix}$$

where $s \in \mathbb{R}^*$ and $u \in \mathbb{R}$. Determine both a left and a right invariant integral, and verify that they are different.

PROBLEM 11. *(Euler angles).*Let α and β be the two one-parameter groups of SO(3) given by

$$\alpha(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \beta(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

Let $\gamma\colon \mathbb{T}^3 \to \mathrm{SO}(3)$ be the map $(s,t,u) \mapsto \alpha(s)\cdot\beta(t)\cdot\alpha(u)$ — where $0 \leq s,t,u \leq 2\pi$. Show that γ is surjective and describe the fibres of γ. The angles s,t,u are called Euler angles.

Show that the normalised, invariant integral on SO(3) can be expressed as

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin t \cdot f(\gamma(s,t,u))\, ds\, dt\, du$$

PROBLEM 12. Show that

$$\exp(v+w) = \lim_{n\to\infty} \left(\exp\left(v/n\right)\exp\left(w/n\right)\right)^n$$

for any vectors v, w in the Lie algebra $\mathrm{Lie}\, G$ of a Lie group G.

PROBLEM 13. Show that the image of the exponential map is a union of conjugacy classes.HINT: Use that $\exp \mathrm{Ad}_g v = c_g(\exp v)$ where $c_g(x) = gxg^{-1}$.

Versjon: Monday, September 17, 2012 2:49:33 PM