Ark2: Exercises for MAT4270 — Basics, Specific groups, π_0 and π_1.

THIS SHEET CONCERNS THE WEEK AUGUST 27 TO AUGUST 31

Plans are approximatly the following:

On Tuesady I did: Specific groups.

On Friday I plan to do: Finish specific groups for this time, that is do the symplectic case. Start on coverings and connected components.

Next week: Finish con-comps and coverings. Start on Lie algebras, invariant vector fields, exponential maps.

Exercises

QUATERNIONS

PROBLEM 1.

a) Show that the centre of the quaternions $\mathbb{H}$ is the set $\mathbb{R}$ of real quaternions.

b) Show that if $e \in \operatorname{Im}\mathbb{H}$, then the centraliser of e, that is the set of elements commuting with e, equals $\{\, x + ye \mid x, y \in \mathbb{R} \,\}$.

c) Let $e \in \mathbb{H}$ be a pure imaginary norm one quaternion, and let $q = x + ye$ be of norm one. Show that $q^{-1} = x - ye$.

d) Show that there are univeral polynomials $P_n(X, Y)$ and $Q_n(X, Y)$, homogenous of degree n and whith integral coefficients, such that for any integer $n \in \mathbb{Z}$

$$q^n = P_{|n|}(x, y) + Q_{|n|}(x, y)e$$

where $q = x + ye$ is a quaternion of norm one.

PROBLEM 2. Show that $\mathbb{H}^* \simeq \mathbb{R}^+ \times \mathrm{SU}(2)$ as Lie groups.

PROBLEM 3. Let i, j and k be the standard basis for the quaternion algebra. One may identify $\mathbb{C}$ with the subspace $\mathbb{R} + i\mathbb{R}$.

a) Show that then $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, and that one has the rule $jz = \bar{z}j$ for all $z \in \mathbb{C}$ which defines the multiplicative structure.

b) Let V be a complex vector space and assume given an additive map $J\colon V \to V$ satisfying $J(zv) = \bar{z}J(v)$. Show that this gives rise to a structure as a quaternionic vectorspace on V.

SEMIDIRECT PRODUCT

PROBLEM 4. *(Semidirect product I).*Let H and A be two Lie groups. Assume that A acts on H in a smooth way meaning that there is a smooth map $\gamma\colon A \times H \to H$ fullfilling the requirements of an action. We denote the value of γ at (a, h) by $\gamma_a(h)$. Define a product on $H \times A$ by the formula

$$(h, a)(h', a') = (h\gamma_a(h'), aa').$$

a) Show that this defines a Lie group structure on $H \times A$.
The Lie group is called the *semi-direct product* of H and A, and we often denoted it by $H \times_\gamma A$ to remember that the action γ is an essential ingredient.

b) Show that $H \simeq H \times \{e\}$ and $A \simeq \{e\} \times A$ are a strong subgroups of $H \times_\gamma A$.

c) Show that H is a normal subgroup of $H \times_\gamma A$ and that the action of A on H by conjugation is the action γ.

PROBLEM 5. *(The semidirect product II).*Let G be a Lie group. Assume that there are two strong subgroups, $H \subseteq G$ and $A \subseteq G$ and that H is normal.

a) Show that action γ of A on H by conjugation is smooth.

b) Show that if the induced homomorphism of (ordinary) groups $A \to G/H$ is an isomorphism, then G is isomorphic (as a Lie group) to the semi-direct product $H \times_\gamma A$.

SPRCIFIC GROUPS

PROBLEM 6. Let Q is the $2n \times 2n$-matrix

$$Q = \begin{pmatrix} 0 & I_n \\ -I_n & = \end{pmatrix}$$

where I_n denotes the $n \times n$-identity matrix.

a) Show that $Q^2 = -1$ and $Q^{-1} = -Q$.

b) Show that if a is a symplectic matrix, *i.e.*, $a^t tQa = Q$, then the eigenvalues of a come in quadruples: λ, $\bar{\lambda}$,λ^{-1},$\bar{\lambda}^{-1}$. HINT: If $P(\lambda)$ is the characteristic polynomial to a, then $P(\lambda) = \lambda^{2n} P(\lambda^{-1})$.

c) Show that the determinant of a is one.

PROBLEM 7. Let G be a Lie group and let $J\colon G \to G$ be a *homomorphism* of Lie groups. Let H be the set of fixed points of J, that is $H = \{\, g \in G \mid Jg = g \,\}$.

a) Show that H is a subgroup.

b) Show that H is the fibre over e of the map $\Psi(g) = J(g)g^{-1}$. Show that Ψ is smooth, being the composition

$$G \xrightarrow{\mathrm{id}_G;J} G \times G \xrightarrow{\nu} G$$

where as usual $\nu(x, y) = xy^{-1}$.

c) Show that the rank of Ψ is the same everywhere, and that H is a strong Lie subgroup of G.

d) Compute the derivative of Ψ at the unit element, and show that the derivative $j = d_e J$ of J at e is an endomorphism of the tangent space T_eG (*i.e.,* it maps T_eG into it self), and show that

$$T_eH = \{\, v \in T_eG \mid j(v) = v \,\}.$$

PROBLEM 8. Let $G = \mathrm{Sl}(n, \mathbb{C})$. We have two maps $c(a) = \bar{a}$ (conjugate all the entries in a) and $d(a) = ((\bar{a})^{-1})^t$ (conjugate all the entries, invert and transpose a) from G to G.

a) Show that they both are involutive (meaning the square is the identity) homomorphims, and that the groups of fixed points are $\mathrm{SU}(n)$ and $\mathrm{Sl}(n, \mathbb{R})$ respectively.

b) Compute the derivatives of c and d at a fixed point $a \in H$, and show that $d_a d$ and $d_a c$ are antilinear involutions $T_aH \to T_aH$ (where $H = \mathrm{SU}(n)$ respectively $H = \mathrm{Sl}(n, \mathbb{R})$), a map j of complex vectorspaces being antilinear if $j(zv) = \bar{z}j(v)$.

PROBLEM 9. Show that $\mathrm{SO}(n, \mathbb{R}) = \mathrm{SU}(n) \cap \mathrm{Sl}(n, \mathbb{R})$ as subgroups of $\mathrm{Sl}(n, \mathbb{C})$.

PROBLEM 10. Show that $\mathrm{U}(n) = \mathrm{Gl}(n, \mathbb{C}) \cap \mathrm{SO}(2n)$ as subgroups of $\mathrm{Gl}(2n, \mathbb{R})$.

CONNECTED COMPONENTS AND UNIVERSAL COVERS.

PROBLEM 11. Assume that G is a Lie group with $G_0 \simeq \mathbb{S}^1$ and G/G_0 cyclic of odd ordre. Show that G is abelian and $G \simeq \mathbb{S}^1 \times G/G_0$.

PROBLEM 12. Show that $\mathrm{O}(3) \simeq \mathrm{SO}(3) \times \mu_2$.

PROBLEM 13. Let G and H be a Lie groups of the same dimension, and assume that H is connected. Assume further that there is an injective homomorphism $\phi\colon H \to G$. Show that ϕ is an isomorphism between H and the connected component G_0 of the unite $e \in G$.

PROBLEM 14. Let $V = \mathrm{M}_2(\mathbb{C})$ be the space of complex 2×2-matrices. Let

$$q(a,b) = \frac{1}{2}\,\mathrm{tr}(ab^\flat)$$

where $b^\flat$ is the *cofactor* matrix of b — obatined by swapping the diagonal elements and swapping the sign of the two other entries.

a) Show that $q(a,b)$ is a symmetric bilinear form on V with $q(a,a) = \det a$.

If $g, h \in \mathrm{Sl}(2, \mathbb{C})$, then the map $a \mapsto gah^{-1}$ preserves the form q, and hence we obtain a map $\Lambda\colon \mathrm{Sl}(2, \mathbb{C}) \times \mathrm{Sl}(2, \mathbb{C}) \to \mathrm{SO}(4, \mathbb{C})$.

b) Show that Λ is a Lie group homorphism whose kernel is isomorphic with $\mathbb{Z}/2\mathbb{Z}$.

c) Show that Λ is the universal cover of $\mathrm{SO}(4, \mathbb{C})$.

PROBLEM 15. Let $G \subseteq \mathrm{U}(n) \times \mathbb{R}$ be the subset $\{\, (g,t) \mid \det g = e^{2\pi i t} \,\}$.

a) Show that G is a strong subgroup, and that the projection onto $\mathrm{U}(n)$ induces an exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow G \xrightarrow{\ p\ } \mathrm{U}(n) \longrightarrow 1$$

Conclude that p is a covering.

b) Show that there is an exact sequence

$$1 \longrightarrow \mathrm{SU}(n) \longrightarrow G \longrightarrow \mathbb{R} \longrightarrow 0$$

induced from the second projection. Conclude that G is connected and simply connected, and hence $p\colon G \to \mathrm{U}(n)$ is the universal cover of $\mathrm{U}(n)$.

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