Ark3: Exercises for MAT4270 — Lie algebras, π_0 and π_1.

THIS SHEET CONCERNS THE WEEK SEPTEMBER 1 TO SEPTEMBER 5

Plans are approximatly the following:

On Tuesday I did: Finished universal cover. Started on the construction of a Lie algebra structure on T_eG.

On Friday I plan to do: Finish construction of a Lie algebra structure on T_eG,*i.e.*, prove The Jacobi identity. Talk about the adjoint representation. Next week: Lie algebra of subgroups. Invariant vector fields, exponential maps.

Exercises

SPECIFIC GROUPS

PROBLEM 1. The aim of this exercise is to show that $\mathrm{Sl}(2, \mathbb{R})$ maps surjectively to the identity component of $\mathrm{O}(1,2)$. We let g be the norm on $\mathbb{R}^3$ of index $(1,3)$ given by

$$g(x) = x_0^2 - x_1^2 - x_2^2$$

when $x = (x_0, x_1, x_2)^t$.

a) Let $M \subseteq \mathrm{M}_2(\mathbb{R})$ be the subspace of *symmetric matrices*. Any element x from M can be represented in the following way:

$$x = \begin{pmatrix} x_0 + x_2 & x_1 \\ x_1 & x_0 - x_2 \end{pmatrix}.$$

Show that $g(x) = \det x$.

b) Show that if $a \in \mathrm{Sl}(2, \mathbb{R})$, then $axa^t \in M$ whenever $x \in M$, and that $g(ax) = g(x)$.

c) Show that sending a to the map $x \mapsto axa^t$ defines a Lie group homorphism $\phi\colon \mathrm{Sl}(2, \mathbb{R}) \to \mathrm{O}(()1,2)$ whose kernel is $\mu_2 = \{\pm 1\}$.

d) Show that $\mathrm{Im}\,\phi$ equals the identity component $\mathrm{O}(1,2)$.

CONNECTED COMPONENTS AND UNIVERSAL COVERS.

PROBLEM 2. Let G_1 and G_2 be two connected Lie groups and denote by $\pi_i\colon \widetilde{G}_i \to G_i$ their universal covers.
a) Show that any Lie group homomorphism $\phi\colon G_1 \to G_2$ lifts to a unique Lie group homomorphism $\widetilde{\phi}\colon \widetilde{G}_1 \to \widetilde{G}_2$.
b) Give an example where $\widetilde{\phi}$ is injective, but ϕ is not.
c) Show that sending G to $\widetilde{G}$ and ϕ to $\widetilde{\phi}$ is a functor

$$\mathsf{Liegroups} \to \mathsf{Liegroups}_0$$

where $\mathsf{Liegroups}_0$ is the category of connected and simply connected Lie groups.

PROBLEM 3. Let G and H be two Lie groups and $\phi\colon G \to H$ a map of Lie groups.
a) Show that $\phi(G_0) \subseteq \phi(H_0)$, and that sending G to G_0 and ϕ to $\phi_0 = \phi|_{G_0}$ is a functor.
b) Show that ϕ is surjective if and only ϕ_0 is surjective, and that ϕ is injective if and only if ϕ_0 is.

PROBLEM 4. The universal cover of $\mathbb{S}^1$ is the exponential map $\mathbb{R} \to \mathbb{S}^1$ as usual given as $e^{2\pi it}$. Show that there is an exact sequence of Lie groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow 0$$

a) Show that any Lie group map $\phi\colon \mathbb{R} \to \mathbb{R}$ is $\mathbb{R}$-linear, *i.e.,* $\phi(x) = x\phi(1)$ for all x and use this to show that any Lie group map $\mathbb{S}^1 \to \mathbb{S}^1$ is of the form $x \mapsto x^n$ for some n.

PROBLEM 5. Shov that the universal cover of $\mathbb{S}^1 \times \mathbb{S}^1$ is the map

$$\mathbb{R} \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1 \text{ given by } (s,t) \mapsto (e^{2\pi is}, e^{2\pi it}).$$

a) Show that group of Lie group automorphisms of $\mathbb{S}^1 \times \mathbb{S}^1$ is given as

$$\mathrm{Aut}(\mathbb{S}^1 \times \mathbb{S}^1) = \mathrm{Sl}(2, \mathbb{Z}).$$

LIE ALGEBRAS PROBLEM 6. Recall that if A is any algebra over a ring R (not necessarily commutative nor associative) an $\mathbb{R}$-linear map $D\colon A \to A$ is called a *derivation* if for any pair of elements a, b from A it fullfills

$$D(ab) = aD(b) + D(a)b \tag{1}$$

Show that L is a Lie algebra, then for any $x \in L$, the map $ad_x\colon L \to L$ given by $ad_x(y) = [x, y]$ is a derivation; that is it satisfies:

$$ad_x([y, z]) = [y, ad_x(z)] + [ad_x(y), z].$$

PROBLEM 7. Verify that $M_n\mathbb{K}$ is a Lie algebra under the commutator product $[x, y] = xy - yx$; *i.e.,* check the Jacobi identity.

PROBLEM 8. Convince yourself that the only thing you used when you solved problem 7 was the associative law. Hence if A is any associative ring, then A is a Lie algebra with the commutator $[x, y] = xy - yx$ as Lie product.

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