Ark4: Exercises for MAT4270 — Lie algebras.

This sheet concerns the week September 17 to September 21

Plans are approximatly the following:

On Tuesday Sep 11 I did: Left invariant vectorfields, one parameter groups, exponential map.

On Friday Sep 14 I plan to do: Classify abelian Lie groups. Start on Baker-Campbell-Hausdorff formula

Exercises

Specific groups

Problem 1. *(The Heisenberg group).*Let H_3 be the group of upper triangular matrices with all diagonal elements equal to one; that is, the matrices of the form

$$a = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where the entries come from the field $\mathbb{K}$.

a) Show that centre $Z(H_3) \simeq \mathbb{K}$ of H_3 and that there is an extension

$$0 \longrightarrow \mathbb{K} \longrightarrow H_3 \longrightarrow \mathbb{K}^2 \longrightarrow 0$$

where $\mathbb{K}$ and $\mathbb{K}^2$ are the additive groups.

b) Show that the Lie algbra $\operatorname{Lie} H_3$ of H_3 is the set of *strictly upper trtiangular* matrices, *i.e.,* matrices of the form:

$$v = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

c) Let p, q and r be the elements

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

.

Show that $[p, r] = [q, r] = 0$ and that r generate the *centre* of $\operatorname{Lie} H_3$. Show that $[p, q] = r$.

d) Show that

$$\exp v = \begin{pmatrix} 1 & x & y + \frac{1}{2}xz \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

PROBLEM 2. Let

$$a = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

find $\exp a$

CONNECTED COMPONENTS AND UNIVERSAL COVERS.

PROBLEM 3. Let G be the following Lie group. As a manifold it is isomorphic to $\mathbb{C} \times \mathbb{C}$ and the product is given as

$$(z, w) \cdot (z', w') = (z + z', w + e^z w').$$

a) Check that G is a simply connected Lie group. Show that it is the semi direct product of $\mathbb{C} \times_\gamma \mathbb{C}$ where the action γ is $\gamma(z)w = e^z w$. Show that the centre of G is the subgroup $\{ (2\pi ni, 0) \mid n \in \mathbb{Z} \}$.

b) For any $n \in \mathbb{N}$ let G_n be the following Lie group. The underlying manifold is $\mathbb{C}^* \times \mathbb{C}$ and product is given as

$$(z, w) \cdot (z', w') = (zz', w + z^n w').$$

Convince yourself that this is a Lie group, and show that the unicersal cover is the group G above.

c) Show that if, $n \neq m$ then G_n and G_m are not isomorphic.

LIE ALGEBRAS.

PROBLEM 4. Let L be a Lie algebra over $\mathbb{K}$. Let $\mathrm{ad}_\star\colon L \to \mathrm{Aut}(L)$ be the *adjoint representation*, that is $\mathrm{ad}_v w = [v, w]$. (This is an "abstract" version of the $\mathrm{ad}_\star$ for the Lie algebra of a group.) Show that $\mathrm{ad}_\star$ is a homorphism of Lie algebras. That is

$$\mathrm{ad}_{[v,w]} = \mathrm{ad}_v \mathrm{ad}_w - \mathrm{ad}_w \mathrm{ad}_v$$

HINT: The Jacobi identity.

PROBLEM 5.

a) Let L be a nonabelian Lie algebra over $\mathbb{K}$ of dimension 2. Show that there is a basis x, y of L with $[x, y] = x$.

b) Let M be the subset of $\mathrm{Gl}(2, \mathbb{K})$ consisting of matrices of the form

$$\begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix}$$

with s and $t \in \mathbb{K}$. Show that $\mathrm{ad}_\star$ induces an isomorphim between L and M. HINT: Compute the matrices of ad_x and ad_y in standard basis.

c) Let G be the subgroup of $\mathrm{Gl}(2, \mathbb{K})$ whose members are the matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

Show that G is diffeomorphic to $\mathbb{K}^* \times \mathbb{K}$. Is G connected? Is it simply connected? (The answers depend on $\mathbb{K}$). Show that $\mathrm{Lie}\, G \simeq L$.

PROBLEM 6. Let e_{ij} be the $n \times n$-matrix with zeros everywhere except at the place (i, j), *i.e.*, $(e_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$.

a) Show that $e_{ij}e_{jk} = e_{ik}$.

b) Show that the linear span of the three matrices e_{ij}, e_{ji} and $e_{ii} - e_{jj}$ is a Lie algebra which is isomorphic to $\mathrm{sl}(2, \mathbb{K})$.

PROBLEM 7.

a) Give an example of a subalgebra that is not an ideal.

b) Recall that the centre of a Lie algebra L is the set $Z(L) = \{\, v \in L \mid [v, w] = 0 \text{ for all } w \in L \,\}$. Show that $Z(L)$ is an ideal.

PROBLEM 8. Let L be a Lie algebra of dimension n over $\mathbb{K}$.

a) Assume that the centre of L is of dimension at least $n-1$. Show that L is abelian. (Recall that the centre of L is the set $Z(L) = \{\, v \in L \mid [v, w] = 0 \text{ for all } w \in L \,\}$)

b) Show that there are exactly two non-isomorphic Lie algebras satisfying $\dim L = n$ and $\dim Z(L) = n - 2$. HINT: If v and w are vectors linearly independent mod the centre, either $[v, w]$ is central or not.

THE EXPONENTIAL MAP.

PROBLEM 9. Show that $\exp\colon \mathrm{sl}(2, \mathbb{C}) \to \mathrm{Sl}(2, \mathbb{C})$ is surjective.

PROBLEM 10. Show that if N is nilpotent (*i.e.,* $N^k = 0$ for som k) then $\exp N$ is unipotent (*i.e.,* $I + M$ for some nilpotent matrix M).

PROBLEM 11. Let v be a vector field on a Lie group G, and let D_v be the global derivation associated to it. Define an action of G on $C^\infty(G)$ by letting $l_g(f) = f \circ \lambda_{g^{-1}}$. Show that v is left invariant if and only if

$$D_v \circ l_g = l_g \circ D_v$$

for all $g \in G$.

PROBLEM 12. Assume that a is an $n \times n$-matrix.

a) Assume that a satisfys $a^2 = -I_n$. Show that

$$\exp ta = \cos t \cdot I_n + \sin t \cdot a.$$

b) Assume that a satisfys $a^2 = I_n$. Show that

$$\exp ta = \cosh t \cdot I_n + \sinh t \cdot a.$$

c) Genetralise the two preciding points by finding a closed formula for $\exp ta$ whenever $a^2 = -dI_n$ for some $d \in \mathbb{K}$.

PROBLEM 13. Show that if $[v, w]$ is central, then $\exp v \exp w = \exp(-[v, w]) \exp v \exp w$.

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