Ark6: Exercises for MAT4270 — Finite groups, Maximal tori

THIS SHEET CONCERNS THE SEVERAL WEEKS

Finite groups

OPPGAVE 1. Assume that the finite group G acts on the finite set X. Let $\mathrm{Fix}(g) = \{ x \in X \mid gx = x \}$ be the fixed point set of the group element g, and recall that G_x denotes the isotropy group of x,*i.e.,* $G_x = \{ g \mid gx = x \}$. Let k denote the number of orbits G has in X.

a) Show that $\sum_{g\in G} \#\mathrm{Fix}(g) = \sum_{x\in X} |G_x|$
HINT: Look at the subset $\{ (g, x) \mid gx = x \}$ of $X \times X$ and use the two projections.

b) Let χ be the character of the permutation representation induced by the action. Show that $\langle \mathbb{1}_G, \chi \rangle_G = k$, and hence that the multiplicity of $\mathbb{1}_G$ in the permutation representation is k.

c) Assume that action of G on X is transitive. Let $x \in X$. Let r denote the number of orbits the isotropy group G_x has on X. Show that r does not depend on x and that $r = \langle \chi, \chi \rangle_G$. HINT: Count the number of orbit that G has on the product $X \times X$.

d) We say that the action of G is doubly transitive, if there for every pair of pairs (!) (x, y) and (x', y') of elements from X there is a group element with $gx = x'$ and $gy = y'$. Show that G that action is doubly transitive if $\chi - \mathbb{1}_G$ is an irreducible character.

In these exercises we adopt the following terminology. A G-module V is said to be *monic* if every irreducible G-module occures in V with multiplicity *at most one* — *i.e.,* if an irreducibel W is summand in V, it is unique.

OPPGAVE 2. Let G be a group and let V be a finite dimensional representation of G over $\mathbb{C}$. Assume that V is completely reducible (*e.g.,* G finite or compact). Show that the algebra $\mathrm{End}_G(V)$ is commutative if and only V is monic.

OPPGAVE 3. Let $H \subseteq G$ be a finite group with a subgroup and let V be a finite dimensional, complex G-module. Show that $\mathrm{End}_H(V)$ is abelian if and only if $\mathrm{res}^G_H V$ is *monic.*

OPPGAVE 4. Let G be a group. Recall that one may identify the group algebra $\mathbb{K}[G]$ with the vector space of functions $f\colon G \to \mathbb{K}$ with f corresponding to $\sum_{x\in G} f(x)x$.

a) Show that the product in $\mathbb{K}[G]$ corresponds to the *convolusion product* of functions:

$$a \star b(g) = \sum_{y\in G} a(y^{-1}g)b(y)$$

b) Let f be a function G with values in $\mathbb{K}$. Let f^t be defined by $f^t(g) = f(g^{-1})$. Show that this induces an anti-involution on $\mathbb{K}[G]$ shuch that $x^t = \sum_g a_g g^{-1}$ if $x = \sum_g a_g g$.

OPPGAVE 5. Let S_n be the group of permutation of $\{1, 2, 3, \ldots, n\}$ and let $S_{n-1} \subseteq S_n$ be the subgroup fixing the the last element n. Show that any element $\sigma \in S_n$ is conjugate to its inverse by an element in S_{n-1}.
HINT: Show that it is enough to check it for cycles. The following might be usefull:If σ is a permutation with $\sigma(i) = s_i$, then $\sigma \circ (1, 2, \ldots, n) \circ \sigma^{-1} = (s_1, s_2, \ldots, s_n)$.

OPPGAVE 6. Construct the character table of the alternating group A_5.

OPPGAVE 7. Let $H \subseteq G$ be a subgroup of the finite group G. Let W be an H-module of finite dimension.

a) Show that $\dim \operatorname{ind}_H^G W = [G : H] \dim V$.

b) Let $\mathbb{K}[G/H]$ be the permutation representation induced from the left action of G on G/H. Show that $\operatorname{ind}_H^G \mathbb{1}_K = \mathbb{K}[G/H]$.

c) What is the character of $\mathbb{K}[G/H]$?

OPPGAVE 8. Let $H = \langle \sigma \rangle \subseteq D_{2n}$ be the subgrup generated by the involution σ in the dihedral group D_{2n}. Compute the character of the induced representation $\operatorname{ind}_H^{D_{2n}} L_{-1}$, where σ acts on L_{-1} by multiplication by -1.

Maximal tori

OPPGAVE 9.

a) Show that the subgroup of $\mathrm{SO}(n)$ with elements of the form

$$\begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_n \end{pmatrix}$$

where $\epsilon_i \in \mu_2$ is a maximal abelian subgroup.

b) Show that any maximal torus in a compact, connected group is a maximal abelian subgroup

OPPGAVE 10. Let G be a compact, connected Lie group.

a) Assume that a maximal torus T in G is normal. Show that $T = G$.

b) Show that any abelian, normal subgroup of G is central.

OPPGAVE 11. Let $g \in G$, a compact, connected Lie group. Show that the identity component of the centraliser $C_G(g)$ is equal to the union of the maximal tori containing g. Give an example that the centraliser $C_G(g)$ is not connected. HINT: Look for elements of $\mathrm{SO}(3)$.

OPPGAVE 12. Let $H \subseteq G$ be a closed subgroup containing the normaliser $N_G T$ of a maximal torus T. Show that $N_G H = H$.

OPPGAVE 13. Let G be a compact, connected Lie group. Show that if $\dim G \leq 2$, then G is a torus.

OPPGAVE 14. Assume that G is compact and connected. If $\dim G = 4$ and $\mathrm{rk}\, G = 2$, show that the center $Z(G)$ either is $\mathbb{S}^1$ or $\mathbb{S}^1 \times \mu_2$. Show that $G/Z(G)$ is isomorphic to $\mathrm{SO}(3)$.

Versjon: Thursday, October 18, 2012 4:18:07 PM