Notes 10: Consequences of Eli Cartan's theorem.

Version 0.00 — with misprints,

The are a few obvious, but important consequences of the theorem of Eli Cartan on the maximal tori. The first one is the observation that all maximal tori of a compact group have the same *dimension*, and this common dimension is called the *rank* of the group. It is always a lot smaller than dimension of the group which considerably facilitates the study of groups. The famous E_8 for example has rank 8 but is of dimension 248!

The second observation is the following

Proposition 1 *If G is a compact, connected Lie group, then the exponential map is surjective.*

PROOF: Since any $g \in G$ is contained in a maximal torus, and the exponential map of a torus is surjective, it follows. ❑

This is certainly not true in general as we saw, $\mathrm{Sl}(2, \mathbb{R})$ is a group with exponential map not being surjective. In fact, one can prove—at least for compact groups—that the exponential map being surjective implies Cartan's theorem about the maximal tori. Another easy consequence of Cartan's theorem is the following:

Proposition 2 *The set of elements of finite order in a compact group G, form a dense subset.*

PROOF: This is cerainly true for the circle and therefore for any torus. If $g \in G$, we may find a torus containing g, and hence a sequence of elements of finite order converging to g. ❑

What follows is a lot more substantial then the two preliminary skirmishes. It is about the structure of the centralisers of abelian subgroups. These centralisers, at least a certain selection of them, play a decisive role in the further theory, and a lot of the structure of the group is contoled by those designated centralizers.

Proposition 3 *Let G be a compact Lie group. If $A \subseteq G$ is any abelian, connected subgroup, then the centralizer $C_G A$ is the union of the maximal tori containing A, i.e., $C_G A = \bigcup_{\text{Tori } T,\ A \subseteq T} T$. In particular, if T is a maximal torus, then $C_G T = T$.*

PROOF: Assume that $x \in G$ centralizes A. Since $C_G\overline{A} = \overline{C_G A}$, we may well assume that A is closed. The closure H of the subgroup generated by x and A is closed and abelian. Let H_0 be its identity component. Then H/H_0 is a finite group B that is generated by the coset xH_0. By lemma 1 below, H has a topological generator, say t, and by Cartan's theorem, t is is contained in a maximal torus. But then the same holds true for the whole of H, and hence for x. ❑

Lemma 1 *Let H be a compact, abelian Lie group with identity component H_0, and assume that H/H_0 is a cyclic group. Then H has a topological generator.*

PROOF: Let $x \in H$ be such that the coset xH_0 generates H/H_0. Since H/H_0 is a finite group, $x^m \in H_0$ for some m. Let $t \in H_0$ be a topological generator, and since any torus is divisible, there are elements $y \in H_0$ with $y^m = t$. For any of those, yx will be a topological generator for H. ❑

Recall that the *centre* $Z(G)$ of G is the subgroup of elements commuting with every element in G. As a members g of the centralizer centralizes any torus, it follows from the proposition that g is contained in any maximal torus. On the other hand, if g lies in every maximal torus, it must commute with any $x \in G$, since by Cartan's theorem x belongs to at least one maximal torus. Hence we have:

Proposition 4 *The centre $Z(G)$ of the compact, connected Lie group G consists of the elements lying in every maximal torus, i.e., $Z(G) = \bigcap_{T \subseteq G} T$ where T runs through the maximal tori.*

EXAMPLE 1. — THE CENTRE OF U(n) AND SU(n). The centre of the unitary group $\mathrm{U}(n)$ is the circle $\mathbb{S}^1$, *i.e.,* the subgroup $\{\, \lambda \cdot \mathrm{id}_{\mathbb{C}^n} \mid |\lambda| = 1 \,\}$. Indeed, any vector in $\mathbb{C}^n$ is the simple eigenspace of some element from $\mathrm{U}(n)$, hence if g is central, g has v as eigenvector. Thus g is a scalar matrix $\lambda \cdot \mathrm{id}_{\mathbb{C}^n}$. Since g is unitary, $|\lambda| = 1$, and $Z(\mathrm{U}(n)) \approx \mathbb{S}^1$.

In the case of $\mathrm{SU}(n)$, the reasoning is the same, except that the determinant of g must be one. Hence $\lambda^n = 1$, and $Z(\mathrm{SU}(n)) \simeq \mu_n$, the group of n-th roots of unity. ❄

EXAMPLE 2. — THE CENTRE OF THE ORTHOGONAL GROUPS. Any two-dimensional subspace of $\mathbb{R}^n$ can be realized as one of the two-dimensional invariant subspaces of a maximal torus in $\mathrm{O}(n)$. Hence if g is in the centre of $\mathrm{O}(n)$, any two-dimensional subspace is invariant under g, and, as long as $n \geq 3$, we see that $g = \lambda \cdot \mathrm{id}_{\mathbb{R}^n}$ for some real number λ—indeed, if $n \geq 3$ any one-dimensional subspace is the intersection of two-dimensional subspaces. Since g is orthogonal, $g = \pm\mathrm{id}_{\mathbb{R}^n}$.

This shows that $Z(\mathrm{O}(n)) \simeq \mu_2$, independently of the parity of n. However for $-\mathrm{id}_{\mathbb{R}^n}$ to have determinant one, n must be even. Therefore $Z(\mathrm{SO}(2m)) = \mu_2$ and $\mathrm{SO}(2m+1) = \{\mathrm{id}\}$. Of course in case $n = 2$, as we know, $Z(\mathrm{SO}(2)) = \mathbb{S}^1$. ❋

Example 3. — The centre of the symplectic groups $\mathrm{Sp}(2m)$. As in the previous examples, any one dimensional subspace of $\mathbb{C}^{2m}$ can be realized as an eigenspace of a subtorus of $\mathrm{Sp}(2m)$. This means that elements in the centre are scalar matrices, and being symplectic, they are of the form $\pm\mathrm{id}_{\mathbb{C}^{2m}}$. Thus $Z(\mathrm{Sp}(2m)) \simeq \mu_2$. ❋

The conjugacy classes.

The set of conjugacy classes of a group G plays a fundamental role when one undertakes an analysis of its representations. The characters of the representations are all class functions, hence functions on the set of conjugacy classes. We denote that set by conj(G). Since the characters are continuous functions on G, we need a topology on conj(G) to single out the functions on conj(G) corresponding to continuous class functions; the natural choice being the quotient topology. Let $\pi\colon G \to \mathrm{conj(G)}$ denote the map sending a group element g to the conjugacy class where it belongs. The quotient topology on conj(G) has the property that for any $f\colon \mathrm{conj(G)} \to X$, where X is any topological space, the map $f \circ \pi$ is continuous if and only if f is—and that is exactly what we wanted.

Theorem 1 *Let G be a compact Lie group and $T \subseteq G$ a maximal torus. The mapping c sending an element $t \in T$ to its conjugacy class in* conj(G) *induces a homeomorphism*

$$c\colon T/W \to \mathrm{conj(G)}\,.$$

Proof: Restricting the map π above to T gives us c, which therefore is continuous since π is.

The content of the theorem of Eli Cartan is that c is surjective. It is injective if any two elements in T, conjugate in G, in fact are conjugate in the normalizer

$N_G T$, and that is the content of the lemma below. One verifies that both spaces are Hausdorff, and then c is a homeomorphism by general properties of topological spaces. ❑

Lemma 2 *Assume that x and y are elements in T conjugate in G. Then there is an element $w \in N_G T$ with $wyw^{-1} = x$.*

PROOF: Let $g \in G$ be such that $gxg^{-1} = y$. Then the conjugation map c_g induces an isomorphism $C_G(x)$ and $C_G(y)$. Now $T \subseteq C_G(x)$ is a maximal torus, and $c_g T \subseteq C_G(y)$ is therefore a maximal torus in $C_G(y)$. The maximal torus T is contained in $C_G(y)$, and must be conjugate to $c_g T$ in $C_G(y)$. Hence there is an element h centralizing y such that $gTg^{-1} = hTh^{-1}$, that is $w = g^{-1}h \in N_G T$. Using that h centralizes y, one verifies

$$wyw^{-1} = g^{-1}hyh^{-1}g = g^{-1}yg = x.$$

❑

The continuous functions on T/W are just the continuous functions on T that are W-equivariant, *i.e.,* satisfy $f(wtw^{-1}) = f(t)$ for all $w \in W$. The class functions on G are just continuous functions on conj(G). Hence we get the following important corollary where classfu(G) stands for the set of class functions on G.

Corollary 1 *There is an isomorphism*

$$\text{classfu(G)} \approx C^0(T)^W$$

This is a step towards an isomorphism between the representation ring $R_{\mathbb{C}}G$ of G and ring $R_{\mathbb{C}}T^W$ of invariants under the Weyl group in the representation ring of T; and indeed it shows that $R_{\mathbb{C}}G \subseteq R_{\mathbb{C}}T^W$. Furthermore, it follows that any complex character on G comes from a class function on T, but the question if it comes from a *character of a representation* of T, is more subtle, even the weaker question if it comes from a *virtual character* is subtle.

EXAMPLE 4. — VIRTUAL REPRESENTATIONS OF SU(n) AND U(n). In the case of U(n), a maximal torus T is n-dimensional, and the ring of virtual representations of T is $R_{\mathbb{C}}T = \mathbb{Z}[c_1, c_2, \ldots, c_n, c_n^{-1}]$, a polynomial ring in the c_i's localized in the multiplicative system generated by the c_i's. Clearly, if we write $d_n = c_1 c_2 \ldots c_n$ for the product, then

$$R_{\mathbb{C}}T = \mathbb{Z}[c_1, \ldots, c_n, d_n^{-1}].$$

The Weyl group of $\mathrm{U}(n)$ is the full symmetric group S_n acting on T by permutation of the coordinates. This action translates to the action on $R_{\mathbb{C}}T$ permuting the c_i's. Clearly d_n is an invariant element for this action. If we let σ_i be the i-th elementary function in the c_i's, then $\sigma_n = d_n$ and $R_{\mathbb{C}}T^{S_n} = \mathbb{Z}[\sigma_1, \dots, \sigma_n, \sigma_n^{-1}]$, and indeed one has

Proposition 5

$$R_{\mathbb{C}}\,\mathrm{U}(n) = \mathbb{Z}[\sigma_1, \dots, \sigma_n, \sigma_n^{-1}].$$

PROOF: The challenge is to verify that all of the characters σ_j's on T are induced from representations on G, *i.e.,* that they all can be extended to the whole of G. In the present case of $\mathrm{U}(n)$, we can, without to much effort, show that this is the case because there are obvious candidates for those representations, namely the *exterior powers* of the basic representation $V = \mathbb{C}^n$ of $\mathrm{U}(n)$.

Let $v_1, \dots, v_n$ be a basis for V that diagonalizes T. For any subset $I = \{i_1, \dots, i_r\}$ of $[1, n]$ with r elements, $1 \leq r \leq n$, we let $v_I = v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_n}$. In our current context the order is of no importance, but for consistency we assume that $i_1 < i_2 < \cdots < i_r$.

Each v_i is an eigenvector for T with character $c_i(t)$, hence v_I is an eigenvector for the induced action of T on $\overset{r}{\Lambda} V$ whose character is $\prod_{i \in I} c_i$. But, the different v_I's form a basis for $\overset{r}{\Lambda} V$ as I runs through the subsets of $[1, n]$, so the character of $\overset{r}{\Lambda} V$ restricts to $\sum_{I \subseteq [1,r]} \prod_{i \in I} c_i = \sigma_i$ in $R_{\mathbb{C}}T$.

The element $v_1 \wedge v_2 \wedge \dots \wedge v_n$ gives us the determinant σ_n, which must be invertible, hence σ_n^{-1} is the restriction of the character of $\overset{n}{\Lambda} V$. ❑

From this example one easily deduces the corresponding statement for $\mathrm{SU}(n)$. Having in mind that any element in $\mathrm{SU}(n)$ has determinant one, the representation $\overset{n}{\Lambda} V$ is trivial, and therefore $\sigma_n = 1$:

Proposition 6

$$R_{\mathbb{C}}\,\mathrm{SU}(n) = \mathbb{Z}[\sigma_1, \dots, \sigma_{n-1}].$$

❄

Groups of rank one

It is a natural part of a theory to explore the cases when the parameters of the theory are small. Usually one gets examples and it sheds some ligth on the general situation. In our case of maximal tori in compact groups, a first question could be: What if the rank is one? Among the examples of groups we already have seen, SU(2) and SO(3) are of rank one, and those two are, as we shall see, the only ones. In addition to the light shedding, that fact is of great importance in the theory and is repeatedly used.

Theorem 2 *Let G be a compact, connected Lie group of rank one. Then either G is isomorphisc to $\mathbb{S}^1$ or $\dim G = 3$.*

PROOF: Equip the Lie algebra $\operatorname{Lie} G$ with an inner product $\langle v, w \rangle$ invariant under the adjoint action, *i.e.*, $\langle \operatorname{Ad}_g v, \operatorname{Ad}_g w \rangle = \langle v, w \rangle$ for all $v, w \in \operatorname{Lie} G$ and all $g \in G$. If you are a sceptic, convince yourself that such an inner product may be found by averaging any inner product over the group against the Haar measure .

Let $t \in T$ be a topological generator, and let $v \in \operatorname{Lie} T$ be a vector of norm one with $\exp v = t$.

The orbit of v under the adjoint action is contained in the sphere $\mathbb{S}^{n-1} \subseteq \operatorname{Lie} G$, hence there is a continuous map

$$\Phi \colon G \to \mathbb{S}^{n-1}$$

sending g to $\operatorname{Ad}_g v$ (which we *a priori* do not know is surjective).

Since T is one dimensional, the inner product invariant and G connected, we must have $\operatorname{Ad}_t v = v$ for all $t \in T$. The map Φ resembles very much the quotient map $G \to G/T$, and we are going to compare the two, and in fact, the crux of the proof is to show they are equal!

By what we just said $\operatorname{Ad}_{gt} v = \operatorname{Ad}_g \operatorname{Ad}_t v = \operatorname{Ad}_g v$ for $g \in G$ and $t \in T$. Hence Φ is constant on the cosets gT, and we get a continuos map ψ as in the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & \mathbb{S}^{n-1} \\ \downarrow & \nearrow_{\psi} & \\ G/T & & \end{array}$$

We shall establish that ψ is an diffeomorphism:

① The map ψ is injective.

We must show that if $\operatorname{Ad}_g v = \operatorname{Ad}_h v$, then $h^{-1}g \in T$. Let $a = h^{-1}g$, then $\operatorname{Ad}_a v = v$. Since T is its own centraliser by proposition 3 on page 1, it suffices to show that

$h^{-1}g$ centralises the topological generator t of T, but this follows from

$$ata^{-1} = a \exp va^{-1} \stackrel{\star}{=} \exp \mathrm{Ad}_a v = \exp v = t,$$

where the equality marked with a star follows from a standard property of the exponential map.

② The map ψ is surjective, and hence a diffeomorphism.
Both spaces $\mathbb{S}^{n-1}$ and G/T have a right action of G, under which the map ψ is equivariant. Hence, by a standard translation argument, ψ is of constant rank. Since it is injective, the rank is $n-1$ by Sard's theorem. By the inverse function theorem, ψ is therefore an open map, diffeomorphic onto is image, and since G/T is compact, ψ is also closed. So $\psi(G/T)$ is both open and closed, hence equal to $\mathbb{S}^{n-1}$, the sphere being connected.

Finally, since $\dim G \equiv \dim T = 1 \mod 2$, it can not happen that $\dim G = 2$. If $\dim G = 1$, then $G = T$. So assume $\dim G > 3$. . The long exact sequence of homotopy groups from the fibration $G \to G/T$ has a portion looking like:

$$\pi_2\mathbb{S}^{n-1} \longrightarrow \pi_1\mathbb{S}^1 \longrightarrow \pi_1 G \longrightarrow \pi_1\mathbb{S}^{n-1}$$

and as $n > 3$, $\pi_2\mathbb{S}^{n-1} = \pi_1\mathbb{S}^{n-1} = 0$. Hence the inclusion $\mathbb{S}^1 \subseteq G$ induces an isomorphism between the fundamental groups. Now by teh sutjectivity og Φ, there ia an $h \in G$ with $\mathrm{Ad}_h v = -v$. Then $c_h(t) = hth^{-1} = t^{-1}$, and c_h induces multiplication -1 in $\pi_1\mathbb{S}^1$. However, as G is connected, c_h is homotopic to the identity in G, and this contradicts the inclusion $T \subseteq G$ inducing an isomorphism of fundamental groups. ❑

Corollary 2 *If G is a compact, connected Lie group of rank one, then G is isomorphic to either $\mathbb{S}^1$, $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Hence if G is not commutative, its Weyl group W is of order two; i.e., $|W| = 2$.*

PROOF: If G is not the circle, then $\dim G = 3$ by the theorem. By fitting out the the Lie algebra $\mathrm{Lie}\, G$ with an inner product invariant under the adjoint action, one obtains a homorphism $G \to \mathrm{SO}(3)$. The derivative at the unit is $\mathrm{ad}_\star \colon \mathrm{Lie}\, G \to \mathrm{Lie}\,\mathrm{SO}(3) \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}\, G, \mathrm{Lie}\, G)$. We do claim that this derivative is injective, and hence it is an isomorphism since dimensions match. If $\mathrm{ad}_v w = 0$ for all $w \in \mathrm{Lie}\, G$, then

$$w = \exp \mathrm{ad}_{uv} w = \mathrm{Ad}_{\exp uv} w$$

for all real u, and it follows that the one parameter subgroup $\exp uv$ is central, thus contained in T. And since $\dim T = 1$, it must be equal to T. But if a maximal torus is central, the group is commutative by Cartan's theorem.

By a standard translation argument, our map is then a covering, and as SU(2) is the universal cover of SO(3), we conclude that G is isomorphic to either SO(3) or SU(2). ❏