Notes 11: Roots.

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Intro

Notation and teminology

The story about the roots can be very confusing fort the uninitiated, it is a field where terminology and notational convention flourish, and there is a multitude of avatars of the roots, complex roots real roots, dual roots, inverse roots. Not to get lost in this jungle of terminology, one must pay the price of being precise to boredom. So we start this section with a detailed but hopefully clarifying recap of the terminology.

The character: A *character* of a torus is a group homomorphism $\chi\colon T \to \mathbb{C}^*$. As $\mathbb{S}^1$ is the only compact, connected subgroup of $\mathbb{C}^*$, considering χ as a map $\chi\colon T \to \mathbb{S}^1$, does not add to the confusion. If V is a complex, one dimensional representation of T, there is a *canonical*[1] character $\chi\colon T \to \mathrm{Aut}_{\mathbb{C}}(V) = \mathbb{C}^*$.

The weight or the root: A character has a derivative at the unit element $\theta = d_e\chi$, or if you want, this is the Lie-functor applied to χ. The derivative is a linear map $\theta\colon \operatorname{Lie} T \to \operatorname{Lie} \mathbb{S}^1$, and it is canonical.

The real vector space $\operatorname{Lie} \mathbb{S}^1$ is equal to the imaginary axis in $\mathbb{C}$, and after having chosen one of the two imaginary units i or $-i$, one has an identification $\operatorname{Lie} \mathbb{S}^1 = i\mathbb{R}$. The map $\theta\colon T \to i\mathbb{R}$ is simply called the *weight* of the representation. In the case the representation is one occurring in the adjoint action of T on $\operatorname{Lie} G$, it is called a *root.*

The complex weight or complex root: The complexified map $\Theta = \theta \otimes \mathrm{id}_{\mathbb{C}}$ is called a *complex weight* or a *complex root* in the case of the adjoint action.

The real weight or real root: Finally, $\alpha = \frac{1}{2\pi i}\theta$ is called a *real weight* or a *real root.* It is linear map $\alpha\colon \operatorname{Lie} T \to \mathbb{R}$, that is, a $linear functional$ on $\operatorname{Lie} T$. It fits

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[1] Indeed, if V is a one-dimensional vector space, the identity is a basis for $\mathrm{Aut}(V)$, as canonical as can be.

into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N_T \arrow[r] \arrow[d] & \operatorname{Lie} T \arrow[r, "\exp"] \arrow[d, "\alpha"] & T \arrow[r] \arrow[d, "\chi"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r, "e"] & \mathbb{S}^1 \arrow[r] & 1
\end{tikzcd} \tag{✷}$$

where e is the map $e\colon \mathbb{R} \to \mathbb{S}^1$ with $e(t) = e^{2\pi i t}$. We see that α takes integral values on the integral lattice N_T.

The adjoint action

The maximal torus T acts on the Lie algebra $\operatorname{Lie} G$ of G by the *adjoint action*. Under this action $\operatorname{Lie} G$ decomposes into a direct sum of irreducible real T-modules, the trivial part being equal to $\operatorname{Lie} T$ as we saw in Notes 9, proposition 3:

$$\operatorname{Lie} G = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in R^+} V_\alpha \tag{✹}$$

The α's runs through all real weights on T. *A priori* there might be repetitions among them. To avoid even more notation, we adopt the convention that V_α is the isotypic component belonging to the root α. Then the subspace V_α is canonically defined. Notationaly, it turns out to be advantageous to adopt the notation V_α for the isotypic part corresponding to the character $\chi\alpha(t) = e^{2\pi i \alpha(t)}$, even if α is not among the roots, but of course then $V_\alpha = 0$. Later on, proposition 1, we shall see that all the α are of multiplicity one.

As $V_\alpha \simeq V_{-\alpha}$ there is an ambiguity in the indexing of components in the decomposition ✹, but as will see, the ambiguity disappears when we complexify the algebra. We let R denote the set of all the real roots that are involved in the decomposition, including both of the opposite roots α and $-\alpha$. If one root from each opposite pair is singled out, in one way or another, we denote the resulting set by R^+.

The *complex Lie algebra* of G is just the complexified Lie algebra $\operatorname{Lie}_{\mathbb{C}} G = \operatorname{Lie} \otimes_{\mathbb{R}} \mathbb{C}$. It has a decomposition inherited from the one (✹) above:

$$\operatorname{Lie}_{\mathbb{C}} G = \operatorname{Lie}_C H \oplus \bigoplus_{\alpha \in R^+} (M_\alpha \oplus M_{-\alpha}). \tag{✻}$$

There is of course a tight relationship between the real modules V_α and the complex ones. Indeed, the complexification $V_\alpha \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as $M_\alpha \oplus M_{-\alpha}$, and V_α is the real part of $M_\alpha \oplus M_{-\alpha}$, that is $V_\alpha = (M_\alpha \oplus M_{-\alpha}) \cap \operatorname{Lie} G$. The characters of the two T-modules M_α and $M_{-\alpha}$ are denoted by χ_α and $\chi_{-\alpha}$ respectively, the relation to the real roots $\pm\alpha$ being that $\chi_{\pm\alpha}(t) = e^{\pm 2\pi i\alpha(t)}$.

The infinitesimal version of the adjoint action of the maximal torus T on $\operatorname{Lie} G$, is the action of $\operatorname{Lie} T$ on $\operatorname{Lie} G$ given by $\operatorname{ad}_\star$; that is, an element $v \in \operatorname{Lie} T$ acts as the endomorphism $w \mapsto [v,w]$. The Jacobi identity guarantees that this is an action Lie algebras:

$$[[v,u],\star] = [v,[u,\star]] - [u,[v,\star]].$$

The relation between the action of T and the infinitesimal action of $\operatorname{Lie} T$ is expressed in the following lemma:

Lemma 1 *Let α be a real root. For each non-trivial $v \in \operatorname{Lie} T$ the isotypic component M_α of $\operatorname{Lie}_{\mathbb{C}} G$ is the subspace such that $[v,w] = 2\pi i\alpha(v)w$ for all $w \in M_\alpha$—that is the eigenspace for $[v,-]$ with eigenvalue $2\pi i\alpha(v)$.*

PROOF: By definition, $\operatorname{Ad}_h w = \chi_\alpha(h)w$ for all $h \in T$ and all $w \in M_\alpha$, that is

$$\exp \operatorname{ad}_{tv} w \overset{\star}{=} \operatorname{Ad}_{\exp tv} w = \chi_\alpha(\exp tv)w = e^{2\pi i t\alpha(v)} w$$

for all $v \in \operatorname{Lie} T$ and all $t \in \mathbb{R}$, where the equality marked with a star is one of the basic formulas for the exponential map (see proposition 6 on page 9 in Note 4). Taking the derivative at zero with respect to the real variable t, we get the lemma since $\exp \operatorname{ad}_{tv} w = w + t[v,w] + o(t^2)$, and $e^{2\pi i t\alpha(v)} = 1 + 2\pi i t\alpha(v) + o(t^2)$. ❑

An extremely useful relation is the following:

Lemma 2 *Let α and β be two real roots. If $a \in M_\alpha$ and $b \in M_\beta$, then $[a,b] \in M_{\alpha+\beta}$.*

PROOF: This is a direct consequence of the Jacobi identity. Let $v \in \operatorname{Lie} T$, $a \in M_\alpha$ and $b \in M_\beta$. We have

$$\begin{aligned}[v,[a,b]] &= -[b,[v,a]] - [a,[b,v]] = 2\pi i\alpha(v)[a,b] + 2\pi i\beta(v)[a,b] = \\ &= 2\pi i(\alpha(v)+\beta(v))[a,b].\end{aligned}$$

❑

Kernels of the characters

The characters χ_α occurring in the decomposition of the complex Lie algebra $\operatorname{Lie}_{\mathbb{C}} G$ are group homomorphism $\chi_\alpha \colon T \to \mathbb{S}^1$, and they play a fundamental role in what follows. Their kernels are denoted by U_α. These kernels are subgroups of T whose connected components are subtori of codimension one, but they do not need be connected (see example 1 below). The Lie algebras $H_\alpha = \operatorname{Lie} U_\alpha$ form a system of hyperplanes in $\operatorname{Lie} T$, a system being a decisive ingredient in the geometric and combinatorial set up called the root system of G. The interplay between these hyperplanes and the action of the Weyl group on $\operatorname{Lie} T$, did turn out to be extremely fruitful in the analysis of Lie groups.

EXAMPLE 1. — IN SU(2) THE U_α ARE NOT CONNECTED.. Let $G = \mathrm{SU}(2)$, and let T be the maximal torus with elements

$$h = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$$

where $z \in \mathbb{S}^1 \subseteq \mathbb{C}$. The Lie algebra su(2) consists of the anti-hermitian matrices, so

$$v = \begin{pmatrix} 0 & a \\ -\overline{a} & 0 \end{pmatrix}$$

is there. One computes

$$\mathrm{Ad}_h v = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & a \\ -\overline{a} & 0 \end{pmatrix} \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^2 a \\ -\overline{z}^2 \overline{a} & 0 \end{pmatrix}$$

So for both the two roots of SU(2), the kernel U_α will be μ_2. ❉

The fundamental relation $e^{2\pi i \alpha(v)} = \chi_\alpha(\exp v)$, shows that the Lie algebra $\operatorname{Lie} U_\alpha$ equals the kernel of α. Digaramholics can enjoy the following commutative diagram that has exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 1 \arrow[d] & \\
 & & H_\alpha \arrow[r, "\exp"] \arrow[d] & U_\alpha \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & N_T \arrow[r] \arrow[d] & \operatorname{Lie} T \arrow[r, "\exp"] \arrow[d, "\alpha"] & T \arrow[r] \arrow[d, "\chi"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r, "e"] & \mathbb{S}^1 \arrow[r] & 1
\end{tikzcd} \qquad (\ast)$$

Reflections

Recall that if $H \subseteq L$ is a hyperplane in a real vector space L, then a *reflection* through H is a non-trivial linear involution $s\colon L \to L$ such that $s(v) = v$ for all $v \in H$. As s is of finite order, in fact two, it is a semi simple endomorphism and consequently there is a basis for L consisting of eigenvectors for s. The vectors from H are all eigenvectors with eigenvalue 1, and in addition, there is a one dimensional eigenspace H^- with the eigenvalue -1.

Later on we shall for each root $\alpha \in R$ exhibit an element s_α from the Weyl group acting as a reflection through the hyperplane H_α. These reflections play a prominent role in the theory. One elementary but very useful result about reflections is the following.

Lemma 3 *Assume that α is linear functional on the vector space L and that s_α is a reflection through the kernel $H_\alpha = \operatorname{Ker}\alpha$. Then there is a unique vector $v_\alpha \in L$ such that*

$$s_\alpha(v) = v - \alpha(v)v_\alpha \tag{$\star$}$$

for all $v \in L$. The vector v_α satisfies $\alpha(v_\alpha) = 2$, and $v_\alpha = v_{-\alpha}$.

PROOF: If $v \in H_\alpha = \operatorname{Ker}\alpha$, then $\star$ is obviously satisfied, s_α being a reflection in H_α.

Let u be one of the vectors in the one-dimensional eigenspace of s_α with eigenvalue -1. We'll determine a scalar a such that au satisfy the equation $\star$. For this to happen, we must have $-u = s_\alpha(u) = u - \alpha(u)au$, giving $0 = (2 - \alpha(u)a)u$, hence $a = 2\alpha(u)^{-1}$, where we use that $\alpha(u) \neq 0$ since $u \notin H_\alpha$. ❑

In the situation in the lemma with a linear functional α and a reflection in the kernel $H_\alpha = \operatorname{Ker}\alpha$, there is a canonically associated dual situation, with the dual space $L^* = \operatorname{Hom}_{\mathbb{R}}(L, \mathbb{R})$ being the space where the reflection takes place.

Any element $v \in L$ can be interpreted as a linear functional $v^\star$ on L^*, namely the functional one could call "evaluation at v", that is, the one defined by $v^\star(\beta) = \beta(v)$. Think of L^* as the vector space we work in and where we want the reflection to take place. The two elements of the pair α and v_α interchange their roles; v_α becomes the linear functional "evaluation at v_α". The kernel is the hyperplane $H_{v_\alpha^\star}$ with members the β's in L^* vanishing on v_α. And α becomes the element. The reflection s_α induces a map $s_{\alpha^*}\colon L^* \to L^*$ by the rule $s_{\alpha^*}(\beta) = \beta \circ s_\alpha$. One easily get the relation

$$s_{\alpha^*}(\beta) = \beta - v_\alpha^\star(\beta)\alpha$$

indeed, just apply β two the relation $\star$ above.

One often uses a scalar product on L such that the reflection s is orthogonal. Then of course v_α is orthogonal to H_α.

Defining the reflections s_α

We now proceed to search for the reflections s_α in the Weyl group of G. The clue is to study the centralizers of the groups U_α. We shall show, in theorem 1 below, that these centralizers are connected (even though U_α is not necessarily connected) and that their Weyl groups all are of order two. As $U_\alpha \subseteq T$, the torus T is contained in the centralizer $C_G U_\alpha$, and in fact it is a maximal torus, being one in G. The corresponding Weyl group of U_α is contained in the Weyl group W of G — the lifting of any of its elements to $C_G U_\alpha$ normalizes T — and being of order two, it gives us the non-trivial involution s_α.

Before starting, we observe that since s_α lifts to an element in the *centralizer* of U_α, it acts trivially on U_α.

It is natural to start the study of centralizers by describing the Lie algebra of the centralizer of an element, that we later will apply to topological generators of various groups.

Lemma 4 *If $x \in T$, then the Lie algebra of the centralizer $C_G(x)$ decomposes as* $\operatorname{Lie} C_G(x) = \operatorname{Lie} T \oplus \bigoplus_{\alpha\in S} V_\alpha$ *where $\alpha \in S$ if and only if $x \in U_\alpha$.*

PROOF: Recall that

$$x(\exp v)x^{-1} = \exp \operatorname{Ad}_x v.$$

It follows that if $\operatorname{Ad}_x v = v$, then $\exp v$ centralizes x. Suppose then that $\exp tv$ centralizes x for all $t \in \mathbb{R}$. Taking the derivatives at $t = 0$ of the two sides of the equation

$$\exp tv = x(\exp tv)x^{-1} = \exp \operatorname{Ad}_x tv = \exp t\operatorname{Ad}_x v,$$

we obtain $v = \operatorname{Ad}_x v$. We have established that $\exp tv$ centralizes x for all t if and only if $\operatorname{Ad}_x v = v$. This means that v lies in the eigenspace of x with eigenvalue one, and that is exactly $\operatorname{Lie} T \oplus \bigoplus_{\alpha\in N} V_\alpha$. ❑

Theorem 1 *Suppose that G is a compact connected Lie group and that $T\subseteq G$ is a maximal torus.*

Let α be one of the real roots of T, and denote by U_α of the character χ_α. Then the centralizer $C_G(U_\alpha)$ is connected and is of dimension $rkG+2$. It has the Lie algebra $\operatorname{Lie} C_G(U_\alpha) = \operatorname{Lie} T \oplus V_\alpha \oplus V_{-\alpha}$, and the Weyl group of $C_G(U_\alpha)$ is of order 2.

PROOF: Denote by U the identity component of U_α. To begin with, we shall establish the theorem for U, and then at the end of proof we will show that $C_G U_\alpha = C_G U$.

So, pick a topological generator u for U. First we remark that $C_G(u) = C_G U$ is a connected group being the union of connected groups with nonempty intersection as described in proposition 3 on page 1 in notes 10. As $u \in U\subseteq T$, the maximal torus T is contained in $C_G(u)$ and of course is a maximal torus there.

The main point of the proof, is to factor out the subgroup U of T to obtain the inclusion $T/U\subseteq C_G(u)/U$, and then observe that $C_G(u)/U$ is of rank one, T/U being a maximal torus. We then apply the theorem about groups of rank one (theorem2 on page 6 in Notes 10), that tells us that $\dim C_G(u)/U = 3$, unless $T/U = C_G(u)/U$. But the latter is not the case, since by lemma 4 both V_α and $V_{-\alpha}$ are contained in $\operatorname{Lie} C_G(u)$, and therefore $\dim C_G(u) \geq \operatorname{rk} G + 2$.

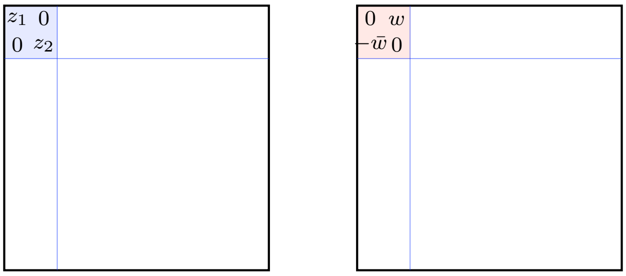
It follows that $\dim C_G(u) = \dim C_G(u)/U + \dim U = \operatorname{rk} G + 2$. From theorem 2 in Notes 10 we also get that the Weyl group of $C_G(u)/U$ is of order 2, but this Weyl group is the same as the one of $C_G(u)$, U being contained in the normalizer of T.

Rests the claim that $C_G U_\alpha = C_G U$. The salient point is that any closed subgroup of T of codimension one, has a topological generator. Indeed, if A is such a subgroup and A_0 denotes the identity component, then A/A_0 is a finite subgroup of $T/A_0 \approx \mathbb{S}^1$, but any finite subgroup of $\mathbb{S}^1$ is cyclic, and we conclude by lemma 1 on page 2 in Notes 10.

To finish the proof, let u be a topological generator of U_α. Then $C_G U_\alpha = C_G(u)\subseteq C_G U$. By lemma 4 $\dim C_G u \geq \operatorname{rk} G + 2$, which implies the desired equality as $C_G U$ is connected and of dimension $\operatorname{rk} G + 2$. ❑

An important consequence of the above reasoning is the following saying that the real roots are simple and that no other root than $-\alpha$ is proportional to α.

Proposition 1 *In the above setting, the representations V_α and $V_{-\alpha}$ occurs just once in the decomposition of $\operatorname{Lie} G$ in irreducible T-modules. Furthermore for any real number r not equal ± 1, the functional $r\alpha$ is not among the real roots.*



Figur 1: The centralizing the 2×2-block and the factor $V_\alpha \oplus V_{-\alpha}$ in $\operatorname{Lie} \mathrm{U}(n)$.

PROOF: By lemma 4 the isotypic components of both V_α and $V_{-\alpha}$ are entirely contained in $\operatorname{Lie} C_G U_\alpha$, but as $\dim C_G U_\alpha = \operatorname{rk} G + 2$, they are both of dimension one. If $r\alpha$ were a real root, then $U_\alpha \cap U_{r\alpha} \neq \emptyset$, and by lemma 4 the representation $V_{r\alpha}$ would be contained in $\operatorname{Lie} C_G U_\alpha$ which is impossible unless $V_{r\alpha} \simeq V_{\pm\alpha}$, that is $r = \pm 1$. ❏

EXAMPLE 2. — $\mathrm{U}(n)$. It might be helpful to have an example in mind when reading, and in this case the simplest case of $\mathrm{U}(n)$ is illustrative. Recall that a maximal torus T of $\mathrm{U}(n)$ is the set of diagonal matrices $D(x)$ with exponentials of the form $e^{2\pi i x_i(x)}$ along the diagonal, where $x = (x_1, \ldots, x_n)$. As long as the all the diagonal elements are different, the centralizer of $D(x)$ is T itself, but at the moment when two become equal, $D(x)$ is also centralized by the corresponding full 2×2-block. For example if $e^{2\pi i x_1} = e^{2\pi i x_2}$, the upper left 2×2-block will centralize $D(x)$. This subgroup is isomorphic to $\mathrm{U}(2)$, but the quotient $C_{\mathrm{U}(n)} U_\alpha / T$ is isomorphic to $\mathrm{SU}(2)$ since the determinant can be absorbed in T, *i.e.*,any $g \in \mathrm{U}(2)$ may be written $g = \det \sqrt{g} \cdot g'$ with an $g' \in \mathrm{SU}(2)$.

The corresponding roots are $\alpha = x_1 - x_2$ and $-\alpha = x_2 - x_1$, and the characters are $\chi_\alpha = e^{2\pi i (x_1 - x_2)}$ and $\chi_{-\alpha} = e^{2\pi i (x_2 - x_1)}$.

Making this argument for each pair of diagonal elements in T, we can conclude that the roots of $\mathrm{U}(()n)$ are $\pm x_i \pm x_j$ for $1 \leq i < j \leq n$, altogether $2n(n-1)$ roots. ❈

EXAMPLE 3. — $\mathrm{SO}(4)$. The group $\mathrm{SO}(4)$ is of rank two and dimension six, so we are looking for four roots. The group has a maximal torus consisting of the matrices with 2 × 2-blocks of the form

$$\begin{pmatrix} \cos 2\pi x_j & -\sin 2\pi x_j \\ \sin 2\pi x_j & \cos 2\pi x_j \end{pmatrix} \tag{✷}$$

along the diagonal. As usual, to simplify the calculations, we go complex. The complexified Lie algebra $\operatorname{Lie}_{\mathbb{C}} \mathrm{SO}(4)$ consists of anti symmetric complex 4 × 4-matrices. They can naturally be divided in two categories:

$$X = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & M \\ -M^t & 0 \end{pmatrix}$$

where those to the left form the subalgebra $\operatorname{Lie} T$ of $\operatorname{Lie} G$, and those to the right constitute the part of $\operatorname{Lie} G$ where T acts non trivially, that is the direct sum of the non-trivial root spaces. The matrix M can be any 2 × 2-complex matrix, and each D_j is of the form

$$D = \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$$

with z replaced by z_j. We may well take them real, and they are linked to the coordinate x_j of $\operatorname{Lie} T$ by $z_j = 2\pi x_j$. The bracket $[Y, X]$ is given by the formula

$$[Y, X] = \begin{pmatrix} 0 & D_1 M - M D_2 \\ -(D_1 M - M D_2)^t & 0 \end{pmatrix}$$

as one readily verifies.

We introduce the two complex matrices

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Then $A, \overline{A}, B, \overline{B}$ is a basis for the part of $\operatorname{Lie}_{\mathbb{C}} G$ where T acts non-trivially, Doing the matrix multiplications, we find that

$$DA = AD = izA \qquad -DB = BD = izB,$$

and using this we get

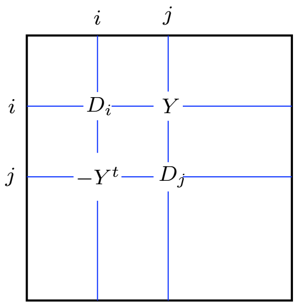
$$D_1 A - A D_2 = 2\pi i (x_1 - x_2) A \text{ and } D_1 B - B D_2 = 2\pi i (x_1 + x_2) B.$$

Conjugating this equation, we have

$$D_1 \overline{A} - \overline{A} D_2 = -2\pi i (x_1 - x_2) \overline{A} \text{ and } D_1 \overline{B} - \overline{B} D_2 = -2\pi i (x_1 + x_2) \overline{B}.$$

Hence the four roots of SO(4) are $\pm x_1 \pm x_2$.

It is not very hard to generalize this to all the special orthogonal groups SO$(2m)$ of even dimension. The maximal torus still has blocks like the ones in ✷ above along the diagonal. The essential computations goes just like what did above, one only has to position the block Y correctly in the matrix, as shown in the figure below where the indices i and j refer to the i-th respectively j-th 2×2-block. One finds that SO$(2m)$ has the $2m(m-1)$ roots $\pm x_i \pm x_j$ for $1 \leq i < j \leq m$. ❋



EXAMPLE 4. — SO(5). The group SO(5) has rank two and dimension ten, hence it has eight roots. It is contained in SO(4), and the two groups share the maximal torus described above. Therefore the roots of SO(4) are also roots of SO(5). In fact, there is an inclusion $\mathrm{Lie}\,\mathrm{SO}(4) \subseteq \mathrm{Lie}\,\mathrm{SO}(5)$, and as the maximal torus is the same, the only difference between the two Lie algebras is two new pair of roots in the bigger one.

To trap those, we introduce the matrices

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & -z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with some effort, one finds after some computations that

$$[D, A] = 2\pi i C.$$

So by positioning the matrices D_i—which are the matrix D with z replaced by $2\pi x_i$—and C correctly in the 5×5-matrix, one sees that the four extra roots are $\pm x_1$ and $\pm x_2$.

The same argument, with the obvious modifications, shows that the special orthogonal groups SO$(2m+1)$ have the roots $\pm x_i \pm x_j$ for $1 \leq i < j \leq m$ and $\pm x_i$ for $1 \leq i \leq m$. There are $2m(m-1)$ of the first kind and $2m$ of the second, altogether $2m^2$. ❋

An integrality property

There are some very strong conditions on the roots that limit the possible root systems considerably. They are really at the hart of the classification of the semi simple Lie groups.

The strongest one is the integrality condition we will establish in this paragraph.

Recall that the kernel of the exponential map $N_T \subseteq T$ is called the integral lattice, and it sits in the usual commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N_T \arrow[r] \arrow[d] & \operatorname{Lie} T \arrow[r, "\exp"] \arrow[d, "\alpha"] & T \arrow[r] \arrow[d, "\chi_\alpha"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r, "e"] & \mathbb{S}^1 \arrow[r] & 1
\end{tikzcd}$$

Proposition 2 *Assume that α is a real root for T. Then the vector v_α lies in the integral lattice; i.e.,$v_\alpha \in N_T$.*

If β is another real root T, then $\beta(v_\alpha) \in \mathbb{Z}$.

PROOF: The last statement follows immediately from the first, in view of the diagram above with α replaced by the root β.

To prove the first, let $w \in \operatorname{Lie} T$ be a the vector with $2w = v_\alpha$ so that $\alpha(w) = 1$. Then $e(\alpha(w)) = 1$, and $\exp w \in \operatorname{Ker} \chi_\alpha = U_\alpha$. Now $\exp(-w) = \exp(s_\alpha w) = s_\alpha \exp w = \exp w$ since s_α acts trivially on U_α. It follows that $\exp v_\alpha = \exp 2w = 1$. $\square$

Root systems

There is an axiomatic description of the root systems that arise in in the theory of Lie groups.

Let V be a real vector space whose elements we will denote by lower case greek letters. The dimension of V is denoted by k, and we assume that V is equipped with an inner product $\langle \alpha, \beta \rangle$.

For any non-zero vector $\alpha \in V$ the hyperplane orthogonal to α will be denoted by H_α, and, of course, it is the kernel of the linear functional $\alpha^* : \beta \mapsto 2 \langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$.

The *reflection* through H_α is the map defined by the equation

$$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{$\ast$}$$

One easily checks that the map s_α has two orthogonal eigenspaces: the hyperplane H_α corresponding to the eigenvalue 1 and the one-dimensional span $\langle\, \alpha \,\rangle$ corresponding to the eigenvalue -1. It follows that s_α is an orthogonal involution.

A finite set R of vectors from V is called a *root system* if the following three criteria are fulfilled:

i) The elements in R span V, and $0 \notin R$.

ii) If $\alpha \in R$ then $-\alpha \in R$, and $-\alpha$ is the only other vector in R proportional to α .

iii) If $\alpha \in R$, then the reflection s_α takes R into R

iv) If $\beta \in r$, then $s_\alpha(\beta) - \beta$ is an integral multiple of α.

The elements of R are called *roots* and the dimension of V is the *rank* of the root system. The subgroup of the orthogonal group $\mathrm{O}(k)$ generated by the reflections is called the *Weyl group* of the root system and denoted W.

It is clear what a map between two roots systems $R \subseteq V$ and $R' \subseteq V'$ should be. It is an orthogonal map $\phi\colon V \to V'$ taking R into R', and, of course, the systems are said to be isomorphic if the map is invertible and induces a bijection between R and R'.

Example 5. — The root system of a Lie group. For any compact, connected Lie group G with a maximal torus T, the corresponding real roots form a root system in the dual $\mathrm{Lie}\, T^*$ of the Lie algebra of G—at least if they span. If that is not the case, they form one in the subspace they span. By virtue of prop xxx, the real roots span $\mathrm{Lie}\, T^*$ if and only if the centre $Z(G)$ is finite.

Indeed, there is an inner product on $\mathrm{Lie}\, T$ invariant under the Weyl group: To get one, as usual, take any inner product and average it over W. This inner product induces an inner product on the dual space $\mathrm{Lie}\, T^*$, by demanding that the dual basis of any orthonormal basis be orthonormal. The first criterion is fulfilled by definition, number ii) is a consequence of proposition 1 on page 7, the third criterion holds true since the reflections s_α are members of the Weyl group and therefore permutes the roots. Finally, prop 2 on page 11 guaranties that the last criterion is satisfied.

❋

For any pair of roots α and β the numbers $n_{\alpha\beta} = 2\,\langle\, \alpha, \beta \,\rangle \,/\, \langle\, \alpha, \alpha \,\rangle$ are integers by the condition iv). They are called the *Cartan numbers* of the root system, and there are very strong restrictions on their values.

Substituting $\langle\, \alpha, \beta \,\rangle = \|\alpha\|\,\|\beta\| \cos\theta_{\alpha\beta}$, where $\theta_{\alpha\beta}$ is the angle between α and β, we obtain

$$n_{\alpha\beta} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta_{\alpha\beta},$$

and consequently there is the relation

$$n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\theta_{\alpha\beta}.$$

Hence $0 \leq n_{\alpha\beta}n_{\beta\alpha} \leq 4$, and if $n_{\alpha\beta}n_{\beta\alpha} = 4$, the angle $\theta_{\alpha\beta}$ is 0 or π, and $\beta = \pm\alpha$. If the product $n_{\alpha\beta}n_{\beta\alpha} < 4$ at least one of the integers $n_{\alpha\beta}$ or $n_{\beta\alpha}$ have to be of absolute value less than one. With out loss of generality, one may assume that $|n_{\alpha\beta}| \leq 1$, and it is then easy to find all possible values of $n_{\beta\alpha}$. They are listed in the following table, where the last row contains the ratio between the square lengths of the roots:

$n_{\alpha\beta}$	0	1	-1	1	-1	1	-1
$n_{\beta\alpha}$	0	1	-1	2	-2	3	-3
$\theta_{\alpha\beta}$	$\pi/2$	$\pi/3$	$2\pi/3$	$\pi/4$	$3\pi/4$	$\pi/6$	$5\pi/6$
	—	1	1	2	2	3	3

We have the following proposition:

Proposition 3 *Assume that α and β are non-proportional roots forming an acute angle, that is $\langle \alpha, \beta \rangle > 0$. Then the difference $\alpha - \beta$ is a root.*

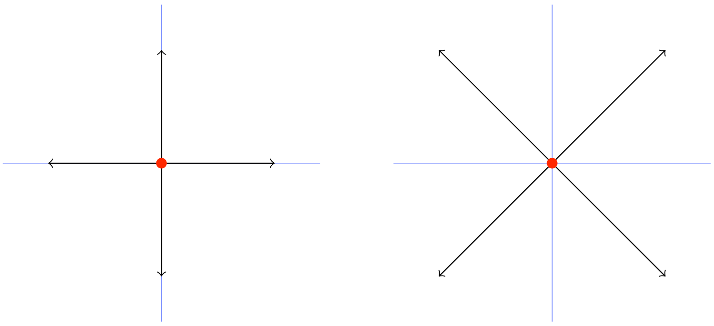
PROOF: As $n_{\alpha\beta} > 0$, either $n_{\alpha\beta} = 1$ or $n_{\beta\alpha} = 1$. In the first case $s_\alpha(\beta) = \beta - \alpha$, and in the other $s_\beta(\alpha) = \alpha - \beta$. ❑

THE PRODUCT OF TWO ROOT SYSTEMS Assume V_1 and V_2 to be two vector spaces each equipped with an inner product. For each $i = 1, 2$, let R_i be a root system in V_i. In the orthogonal sum $V_1 \oplus V_2$, the disjoint union $R_1 \cup R_2$ will be a root system. It is called the *product* of the two systems. Vice versa, given a root system R in V, if there is a disjoint decomposition of R as $R = R_1 \cup R_2$ in mutually orthogonal sets, then V will be equal to the product of the two root system R_i in V_i where V_i is the linear span of R_i. The Weyl group of the product is the product of the Weyl groups of the two root systems. A root system is called *irreducible* if it is not equal to a product of two smaller systems.

EXAMPLE 6. — RANK ONE. Then by property 2, $R = \{e_1, -e_1\}$. Even if simple, this system has a name. It is called A_1 the Weyl group is $\mathbb{Z}/2\mathbb{Z}$. It is the root system of the Lie groups of rank one, and we know there are two, SU(2) and SO(3).

⁂

EXAMPLE 7. — RANK TWO — $A_1 \times A_1$. The number of roots is four and they are $\pm e_1, \pm e_2$. The roots can be split into two orthogonal sets and the system reducible. The Weyl group is the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has this root system.



Figur 2: The root system $A_1 \times A_1$ (left) and $\mathrm{SO}(4)$ (right)

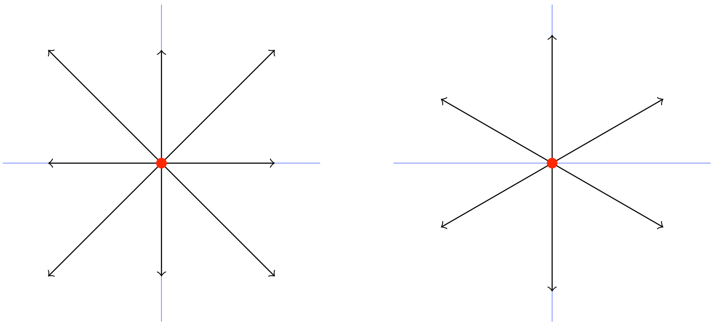
⁂

EXAMPLE 8. — RANK TWO—$\mathrm{SO}(4)$. The number of roots are four: $\pm x_1 \pm x_2$, and as we saw in example 3 these are the roots of $\mathrm{SO}(4)$. The system is decomposable and clearly equivalent to $A_1 \times A_1$. This indicates a strong connection between the two groups $\mathrm{SO}(4)$ and $\mathrm{SU}(2)\times\mathrm{SU}(2)$, and indeed, as we later will show, the universal cover of $\mathrm{SO}(4)$ is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, a relationship often expressed by saying there is an isomorphism $\mathrm{Spin}(4) \simeq \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. ⁂

EXAMPLE 9. — RANK TWO—B_2. This root system has the eight roots $\pm e_1, \pm e_2, \pm e_1 + \pm e_2$. Four short ones, and four long ones, a factor $\sqrt{2}$ longer than the short. Its Weyl group is the Dihedral group D_8, and this is the root system of the orthogonal group $\mathrm{SO}(5)$. ⁂

EXAMPLE 10. The six roots are $\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)$. The Weyl group is the symmetric group S_3 which is the same as the dihedral group D_6, and this is the root system of $\mathrm{SU}(3)$. The roots are drawn in the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. ⁂

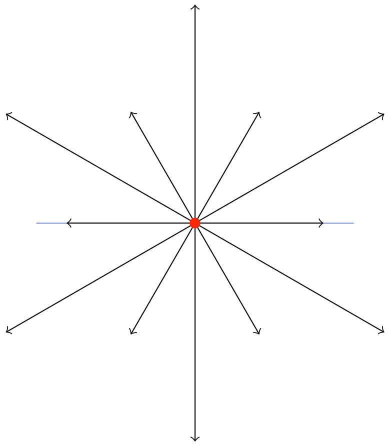
EXAMPLE 11. — RANK TWO— G_2. This root system is an interesting case, being the root system of the the so called *exceptional groups*. The dimension of the group



Figur 3: The two root systems B_2 (to the left) and A_2 (to the right)

G_2 is 14. It is the smallest of the exceptional groups, but even with such a simple root system, the group is not trivial to define. There are 12 roots $\pm(e_1-e_2), \pm(e_1-e_3), \pm(e_2-e_3), \pm(2e_1-e_2-e_3), \pm(-e_1+2e_2-e_3), \pm(-e_1-e_2+2e_3)$, all in the plane $x_1+x_2+x_3=0$. Six of the roots are short and six long, the longer being longer with a factor of $\sqrt{3}$.

One may think about the G_2-system as the superposition of two A_2-systems, but one rotated by an angle $\pi/6$ and scaled by at factor $\sqrt{3}$ compared with the other. The Weyl group is the dihedral group D_6. ❄



Figur 4: The root system G_2

BASES OF ROOT SYSTEMS. Let us fix a linear functional ϕ on V. As we work with a fixed inner product on V, the functional ϕ can be identified with $\phi(\beta) = \langle w_\phi, v \rangle$.

The functional separates V in two to half spaces T^+ and T^- where the functional takes respectively positive and negative values. The frontier between the half spaces is the hyperplane T^0 where ϕ vanishes. The functional ϕ is assumed not to vanish at any root $\alpha \in R$. The roots in the half space T^+ are called *positive roots* and the set they form is denoted by R^+. The counterpart is the set R^- of *negative roots*, that is, those in T^-. As the roots always come in pairs α and $-\alpha$, there are as many positive as negative roots.

It is important to remember the fact that the sets of positive and negative roots depend on the choice of the functional ϕ.

A subset $S\subseteq R^+$ is said to be a *basis* for the root system if the elements of S are linearly independent, and if any β in R can be written

$$\beta = \sum_{\alpha\in S} m_\alpha \alpha$$

where the coefficients m_α are integers either all satisfying $n_\alpha \geq 0$ or all satisfying $n_\alpha \leq 0$. The elements of S are also called *simple roots*, and the reflection s_α is a *simple reflection* if α is a simple root. A root $\alpha \in R^+$ is *indecomposable in* R^+ if it can not be expressed as sum $\alpha = \beta + \gamma$ with α and β in R^+. Every root system has a basis, or more precisely:

Proposition 4 *The set S of indecomposable roots in R^+ is a basis.*

Before giving the proof, we observe that any basis is of this type. Indeed, if a basis is contained in some set of positive roots R^+, then certainly the indecomposable must be members. And as the elements of the basis are linearly independent, they are at most as many as $\dim V$ in number, and there are linear functionals taking prescribed positive values on them. Now to the proof of the proposition:

PROOF: Recall that ϕ denotes the functional that divides V into a positive and a negative half space.

In the finite set of positive roots not being the sum of elements from S there is, if any at all, a root α with $\phi(\alpha)$ minimal. Since $\alpha \notin S$, it is decomposable in R^+, that is, it is expressible as a sum $\alpha = \sum \alpha_i$ with the α_i's from R^+ all different from α. But then the sum $\phi(\alpha) = \sum \phi(\alpha_i)$ has all terms positive, and there is at least two terms. Hence each $\phi(\alpha_i) < \phi(\alpha)$, and by induction, every α_i is a sum of elements from S.

To check that the elements in S are linearly independent, we make use of the following lemma, where the hypothesis of the angle between any pair α and β from

S being obtuse, is fulfilled, since if not, $\alpha - \beta$ is a root by proposition 3 on page 13, and hence either α or β decomposes in R^+. ❏

Lemma 5 *If any two elements of a set* $S \subseteq V$ *make an obtuse angle, and they all lie in the same half space, the elements of* S *are linearly independent.*

Proof: Recall that a half space is the subset of V where a functional ϕ takes on positive values, and that α and β make an obtuse angle, means that $\langle \alpha, \beta \rangle \leq 0$.

A potential dependence relation between elements from the set S may be written as

$$\sum_{i \in I} a_i \alpha_i = \sum_{j \in J} a_j \alpha_j \tag{❁}$$

where the sets $\{\alpha_i\}_{i \in I}$ and $\{\alpha_j\}_{j \in J}$ are disjoint subsets of S, and all the a_i's and a_j's are positive real numbers. If we denote by x the common value of the two sides in equation ❁ we have $\langle \gamma, \gamma \rangle = \sum_{i,j} a_i a_j \langle \alpha_i, \alpha_j \rangle \leq 0$. Hence $x = 0$. On the other hand

$$\phi(x) = \sum_{i \in I} a_i \phi(\alpha_i) > 0,$$

so no dependence relation can be. ❏

Weyl chambers

What we have done so far, is to show that given a linear functional $\phi \in V^*$ not vanishing at any root, we get a uniquely defined basis for the root system: The functional determines the set R^+ of positive roots, and the set of indecomposable is certainly determined by R^+.

If we change the functional, the set of positive roots and the basis will in general be different. However, as long as the set of positive roots remains the same, the basis remains the same. This means that as long as the values $\phi'(\alpha)$ have the same signs as $\phi(\alpha)$ for all $\alpha \in R$, the positive roots and the basis determined by two functionals ϕ' and ϕ are the same. This motivates the definition:

$$K_S = \{ \phi \in V^* \mid \phi(\alpha) > 0 \text{ for all } \alpha \in S \},$$

and this is called *the fundamental Weyl* chamber associated to the basis S.

For any root α in R we let $H_\alpha \subseteq V^*$ denote the hyperplane in V^* consisting of the linear functionals vanishing on α. The finite set R of roots determine a decomposition of V^* into so called *chambers*. The subset of V^* of functionals not vanishing at any of the roots, that is $V^* \setminus \bigcup_{\alpha \in R} H_\alpha$, decomposes into a disjoint union of its connected components, and these components are called the **Weyl chambers**. They are convex cones, open in V^*. The boundary of a Weyl chamber K is the union of closed, convex cones with nonempty interior in some of the hyperplanes H_a. They are called the **walls** of K.

Proposition 5 *Every Weyl chamber is the fundamental Weyl chamber of a unique basis. Hence there is a one-one correspondence between bases and Weyl chambers.*

PROOF: We have already done most of this proof and seen that any basis has a fundamental Weyl chamber. If K is a Weyl chamber, pick any linear functional $\phi \in K$. Then the indecomposable roots in the set of positive roots corresponding to ϕ is a basis with $K_S = K$, and it is the only basis with this property. ❑

The action of the Weyl group

We have seen that a root system has bases and that the bases correspond to the Weyl chambers. The next natural thing to do is to get some understanding of how many bases and chambers there are, and what kind of base changes can take place. The answer lies in the action of the Weyl group:

Theorem 2 *The Weyl group acts freely on the set of Weyl chambers and on the set of bases.*

For the action to be free it must satisfy two criteria. Firstly, it must be transitive, meaning that for any pair of chambers K and L there is an element w of the Weyl group carrying K to L. Secondly, the action must be free for isotropy: The only $w \in W$ fixing a chamber is the identity, *i.e.,*if $wK = K$ for a chamber K, then $w = 1$.

Establishing that the action of the Weyl group does answer to these two demands, requires some effort and some lemmas. The first one is the following which says that if α is a simple root, then the reflection s_α, a part from α it self, permutes the positive roots:

Lemma 6 *If s_α is a simple reflection and β is a positive root different from α, then $s_\alpha\beta$ is a positive root.*

PROOF: We may express $\beta = \sum_{\gamma\in S} n_\gamma\gamma$ where $n_\gamma \geq 0$. As $\beta \neq \alpha$, at least for one $\gamma \neq \alpha$ we have $n_\gamma > 0$, say γ_0. But then in the expression

$$s_\alpha\beta = \beta - n_{\alpha\beta}\alpha = \sum_{\gamma\in S,\gamma\neq\alpha} n_\gamma\gamma + (n_\alpha - n_{\alpha\beta})\alpha$$

the coefficient of γ_0 is n_{γ_0}, and $n_{\gamma_0} > 0$. The set of roots S being a basis, one coefficient being positive implies that all are, and hence $s_\alpha\beta \in R^+$. ❏

Lemma 7 *For any root β there is a simple root α and a sequence of simple reflections carrying α to β*

PROOF: The lemma will be proved for positive roots, it being easy to reduce the general statement to that case. One may write

$$\beta = \sum_{\gamma\in S} n_\gamma\gamma \text{ with } n_\gamma \geq 0. \tag{✻}$$

The proof will be by induction on $h(\beta) = \sum_{\gamma\in S} n_\gamma$. If $h(\beta) = 1$, the root β is simple, and there is nothing to prove. If not, at least two of the coefficients in ✻ are strictly positive, since a nontrivial and positive multiple of a positive root is not a root. Let γ_0 be one of them. Then $h(s_{\gamma_0}\beta) = h(\beta - n_{\gamma\beta}\gamma_0) < h(\beta)$. Furthermore, $s_{\gamma_0}\beta$ is still a positive root since the other strictly positive coefficient in the expression ✻ does not change when the reflection $s_{\gamma_0}$is applied. By induction, there is a simple root α and a sequence of simple reflections whose composition w is such that $w\alpha = s_{\gamma_0}\beta$. Consequently $\beta = s_{\gamma_0}w\alpha$, and we are done. ❏

Proposition 6 *The Weyl group W is generated by simple reflections.*

PROOF: By definition W is generated by reflections s_β for $\beta \in R$. But given β, the lemma gives us a simple root α and a sequence of simple reflections whose composition w is such that $\beta = w\alpha$. Thence $s_\beta = ws_\alpha w^{-1}$. ❏

Proposition 7 *If an element $w \in W$ is such that $wR^+ = R^+$, then $w = 1$*

PROOF: Since W is generated by simple reflections, w can be written as $w = s_r s_{r-1} \dots s_1$ where each s_i is a simple reflection, and where r is minimal, *i.e.,*no such factorization in fewer simple reflections may be found. We have $s_r = s_\alpha$ for a simple root α. Follow the root $\gamma_j = s_j s_{j-1} \dots s_1 \alpha$ as j increases. It starts out in R^+, but at a certain point it is carried into R^-. That swapping is performed by a simple reflection s_γ. We thus may group the s_i's together in such a way to get a factorization

$$s_{r-1} \dots s_1 = a s_\gamma b,$$

where $b\alpha = \gamma$ and $a(-\gamma) = -\alpha$, *i.e.,* $a\gamma = \alpha$. It follows that $a s_\gamma a^{-1} = s_\alpha$, and therefore

$$s_\alpha a s_\gamma b = a s_\gamma s_\gamma b = ab,$$

which contradicts the minimality of r. ❑

This gives us half of theorem 2:

Proposition 8 *The Weyl group W acts without isotropy on the set of bases and on the set of Weyl chambers.*

PROOF: If a basis or a Weyl chamber is stabilized by an element w, the element stabilizes the corresponding set of positive roots, and we conclude by proposition 7 above. ❑

The second half is considerably more easy:

Proposition 9 *The Weyl group acts transitively on the set of Weyl chambers.*

PROOF: Recall that there is an invariant metric on both V and V^*. Let K and L be two Weyl chambers and pick elements $f \in K$ and $g \in L$. Let $w \in W$ be such that $\|wg - f\|$ is minimal, indeed, we find such a w since the Weyl group W is finite[2]. For any wall H_α of K, the elements wg and f must be on the same side of H_α, if not $\|s_\alpha wg - f\| < \|wg - f\|$. Hence $wg \in K$, and as different Weyl chambers are disjoint, it follows that $wL = K$. ❑

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[2] This holds since W embeds into the group of permutations of all the roots.