Notes 12: Dynkin diagrams.

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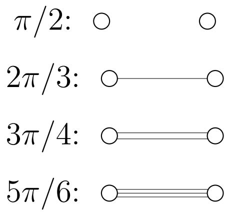
The aim of this paragraph is to defined the so called *Dynkin diagrams* associated to a roots system. There are two slightly different type of diagrams in use, the other type which contains less information about the root system is called *Coxeter diagrams*.

Let a root system with roots $R \subseteq V$ in a vector space V of dimension k be given, and chose a basis S for the system. The angels between the roots in the basis are all obtuse, and as listed in table on page 13 in Notes 11, there are only a few possible value these angles can have, namely the four values $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$. In case the roots are orthogonal, there is no restriction on the ratio between their lengths, but in the three other cases the corresponding ratios are 1, $\sqrt{2}$, $\sqrt{3}$,

We now come to the definition of the *Dynkin diagram*. It is a *graph* with *nodes* and *edges* connecting some of the nodes, where a graph is understood in the wide sense in that several edges connecting the same pair of nodes is allowed[1]. There is to be one node for each element in the basis S, and the number of edges between two nodes depends on the angle the corresponding roots form. If the roots are orthogonal, the nodes are not connected. Two nodes with corresponding roots forming an angle of $\pi/3$ are connected by one edge; they are connected by two edges if the angle is $3/4$ and by three if it is $5\pi/6$.



The diagrams obtained in this way are the *Coxeter diagrams*. In the Dynkin diagrams the different lengths of the roots are also taken into account. That is done by marking the edge connecting two nodes whose corresponding roots are of different length, with an arrowhead pointing to the smaller root. Examples are the two rank two root systems B_2 and G_2, where the long roots in the basis are longer by a factor of respectively $\sqrt{2}$ and $\sqrt{3}$, and the Dynkin diagrams look like:

[1] This is a called a *multigraph* by some.

B_2 G_2

The Dynkin diagram does of course not depend on the chosen basis S. We saw that any two bases are equivalent under Weyl group, and the Weyl group acts by orthogonal transformations. The angles and lengths used in constructing the Dynkin diagram are therefore the same for any two bases.

The Dynkin diagrams of the classical groups

THE GROUPS $\mathrm{U}(n)$ AND $\mathrm{SU}(n)$. These groups are of rank n and $n-1$ respectively. Let T be a maximal torus of $\mathrm{U}(n)$ and let $x_1, \dots, x_n$ be the coordinates on $\mathrm{Lie}\, T$; that is a basis for $\mathrm{Lie}\, T^*$ such that T consists of the diagonal matrices with the exponentials $e^{2\pi i x_j}$ as diagonal elements. The maximal torus of $\mathrm{SU}(n)$ is the kernel of the determinant $\det\colon T \to \mathbb{S}^1$.

We saw that the two groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ have the same roots, namely the linear functionals $\pm(x_i - x_j)$ for $1 \le i < j \le n$. In the case of $\mathrm{SU}(n)$ one easily convinces oneself that a basis for $\mathrm{Lie}\, T^*$ is formed by the $n-1$ roots $\alpha_i = x_i - x_{i+1}$ for $1 \le i < n$. The unitary group $\mathrm{U}(n)$ has the same roots, but it is not semi-simple, the centre of $\mathrm{U}(n)$ being isomorphic to $\mathbb{S}^1$, hence the roots do not span $\mathrm{Lie}\, T^*$.

Using an invariant inner product on $\mathrm{Lie}\, T^*$ scaled in a manner that the x_i's are of norm one, we compute, assuming $i < j$:

$$\langle \alpha_i, \alpha_j \rangle = \langle x_i - x_{i+1}, x_j - x_{j+1} \rangle = -\langle x_{i+1}, x_j \rangle = -\delta_{i+1,j}$$

We conclude that α_i and α_j are orthogonal if $j \ne i+1$, and, the roots all being of the same length $\sqrt{2}$, that α_i and α_{i+1} form an angle of $2\pi/3$. The Dynkin diagram has $n-1$ nodes, is denoted by A_{n-1} and looks like

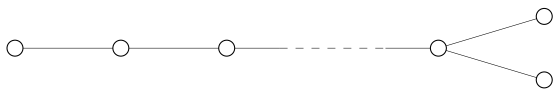
A_{n-1} : 

THE ORTHOGONAL GROUPS $\mathrm{SO}(2m)$. The rank of $\mathrm{SO}(2m)$ is m, and we use coordinates $x_1, \dots, x_m$ on $\mathrm{Lie}\, T^*$, corresponding to the angles appearing in the diagonal blocks of T.

We have shown that the roots of $\mathrm{SO}(2m)$ are are $\pm x_i \pm x_j$ for $1 \le i < j \le n$. A basis is formed by $\alpha_i = x_i - x_{i+1}$ for $1 \le i < m$ and $x_{m-1} + x_m$, as one easily convinces oneself of (with the help of one sum and all the differences, you can generate all the sums). The roots $x_i - x_{i+1}$ behave just as in the previous case, and concerning the new one, we compute

$$\langle\, x_i - x_{i+1}, x_{m-1} + x_m \,\rangle = \begin{cases} -1 & \text{if } i = m-2 \\ 0 & \text{if not }, \end{cases}$$

and $x_{m-1} + x_m$ is orthogonal two all the others except $x_i - x_{i+1}$ with which it forms the angle $2\pi/3$. The Dynkin diagram is denoted by D_m, and it is obtained by adding one new node to A_{m-1} and connecting it to node number $m-2$ by an edge:

D_m : 

THE ORTHOGONAL GROUPS SO$(2m+1)$. The group SO$(2m+1)$ has the same maximal torus as SO$(2m)$. It shares the roots of SO$(2m+1)$, but have gotten $2m$ additional roots. The roots are $\pm x_i \pm x_j$ for $1 \le i < j \le m$, and the additional ones $\pm x_i$ for $1 \le i \le m$. A basis will be $x_i - x_{i+1}$ for $1 \le i < m$ and x_m. The only new thing to compute is

$$\langle\, x_{m-1} - x_m, x_m \,\rangle = -1,$$

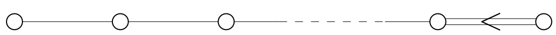
and the length of x_m which is one. It is thus a short root, the old ones having length $\sqrt{2}$ and being long. The angle between $x_{m-1} - x_m$ and x_m is $3\pi/4$. Hence the Dynkin diagram is

B_m: 

THE SYMPLECIC GROUPS Sp$(2m)$ This roots are $\pm x_i \pm x_j$ for $1 \le i < j \le m$ and $\pm 2x_i$ for $1 \le i \le m$. A basis is $x_i - x_{i+1}$ for $1 \le i < m$ and $2x_m$ The only new thing to compute is

$$\langle\, x_{m-1} - x_m, 2x_m \,\rangle = -2,$$

and the length of $2x_m$ which is two. It is thus a long root, the old ones being short, and this is the only thing that distinguishes it from B_m. The angle between $x_{m-1} - x_m$ and x_m is $3\pi/4$. Hence the Dynkin diagram is

C_n : 

Reducible root systems

The Dynkin diagram need not being a connected graph, it might well have several connected components, say $D_1, \dots, D_r$. Two roots from the basis having their nodes in different components are by definition not connected by an edge, hence they are orthogonal. As linear combinations of orthogonal roots are orthogonal, the set of roots R is a disjoint union of the orthogonal subsets $R_1, \dots, R_r$, where the roots in R_i are combinations of basis elements with nodes in D_i. The system is not irreducible. It decomposes as an orthogonal sum $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ with each V_i being the subspace spanned by R_i.

The Dynkin diagram determines the root system

Assume that two root systems $R_1 \subseteq V_1$ and $R_2 \subseteq V_2$ have diagrams D_1 and D_2 with an isomorphism of diagrams $\psi\colon D_1 \to D_2$ between them, where we by an isomorphism of diagrams understand a bijection respecting edges and arrowheads. The nodes of the diagrams correspond to bases S_1 and S_2 in V_1 and V_2 respectively, so ψ gives a bijection between the two bases, hence a linear isomorphism $\Psi\colon V_1 \to V_2$. The angles and lengths are preserved after a scaling, hence ψ induces an orthogonal transformation.

What about the two sets of roots? They are preserved by Ψ due to the transitive action of the Weyl groups on the roots: The map $a \to \Psi a \Psi^{-1}$ clearly induces an isomorphism between the orthogonal groups $\mathrm{O}(V_1)$ and $\mathrm{O}(V_2)$, and as the Weyl groups are generated by simple reflections, that map carries the Weyl group of R_1 into the one of R_2. It follows that Ψ maps R_1 into R_2 since the Weyl groups act transitively on the roots.

Obviously the orthogonal map Ψ is uniquely defined by ψ, but there is a choice of the bases involved, which may be tackled in the following way: If Ψ is any orthogonal map $V_1 \to V_2$ taking R_1 into R_2, then the composition $w' \circ \Psi \circ w^{-1}$, where w and w' are elements from the Weyl groups of V_1 and V_2 respectively, is orthogonal and carries R_1 into R_2. This makes the product of the Weyl groups act on the transformations Ψ.

Any two bases being equivalent under the action of the Weyl group, we get the following proposition:

Proposition 1 *Given $R_1 \subseteq V_1$ and $R_2 \subseteq V_2$ with Dynkin diagrams D_1 and D_2. Then there is a bijection between orbits under the product of the Weyl groups of orthogonal maps $V_1 \to V_2$ carrying R_1 into R_2 and isomorphisms of diagrams $D_1 \to D_2$.*

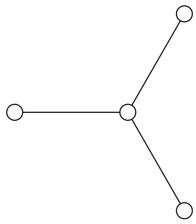
Let $R \subseteq V$ be a root system. The Weyl group $W \subseteq \mathrm{O}(V)$ carries the set of roots R into R, and it is a natural question whether there other subgroups of $\mathrm{O}(V)$ doing the same? By the proposition the answer depends on the root system, it is sometimes yes and sometimes no. For a while, we let $G(R) = \{\, a \in \mathrm{O}(V) \mid aR \subseteq R \,\}$ and $\mathrm{Out}(R) = G(R)/W$.

In the case the of the root system A_n, we have $\mathrm{Out}(A_n) = \mathbb{Z}/2\mathbb{Z}$. Indeed the Dynkin diagram of A_n has a symmetry of order two: If n is even, one may reflect the diagram about the midpoint of the middle edge, and if n is odd, about the middle node. This is the only possible symmetry (the two extreme nodes must be swapped, and this determines the rest of the symmetry).

The two root systems B_n and C_n do not have any symmetries since the arrowhead makes the extreme nodes different. Hence they can not be permuted.

However D_n has symmetries. If $n > 4$, then $\mathrm{Out}(D_n) = \mathbb{Z}/2\mathbb{Z}$: One may exchange the two nodes in the "fork", and that is the only symmetry since the "long tail" must be fixed. If $n = 4$ the diagram has one central node connected to three satellites, which may be permuted at will. Hence $\mathrm{Out}(D_4)$ is the full symmetric group on three letters, *i.e.,*$\mathrm{Out}(D_4) = S_3$.

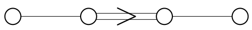
D_4 :



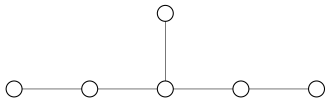
The Killing-Cartan Classification

In beginning of these notes, we described the Dynkin diagrams of the four series of classical groups, the series A_{n-1}, B_m, C_m and D_m. They are all irreducible, in addition there are five so called exceptional root systems: G_2, F_4, E_6, E_7 and E_8. We have already met G_2, and the others are depicted below.

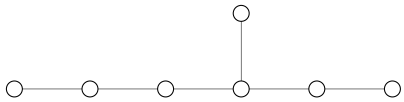
It is a theorem, called the *Killing-Cartan classification*, that these—that is the four classical series and the five exceptional systems—are all the irreducible root systems that exist. Each one of them is the root system corresponding to a Lie algebra, and there are compact connected groups having those Lie algebras.



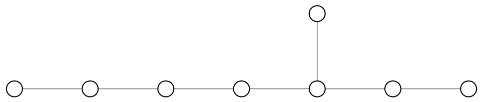
F_4



E_6



E_7



E_8

The Root system determines the Lie algebra.

One may prove that if two Lie groups have the same root systems, then their Lie-algebras are isomophic. It is a rather subtle argument which we do not dive into here.