Notes 8: symmetric groups.

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This note is about the *symmetic groups* S_n of permutations of the set of the first n natural numbers $\{1, \ldots, n\}$, which we by the way shall denote by $[1, n]$ in this section. The representation theory of the symmetric groups is a vast subject , and our modest aim is just to give a short introduction covering a few of the very basic things.

BASICS Recall that any permutation σ can be written as a composition of disjoint cycles

$$\sigma = c_1 c_2, \ldots, c_r.$$

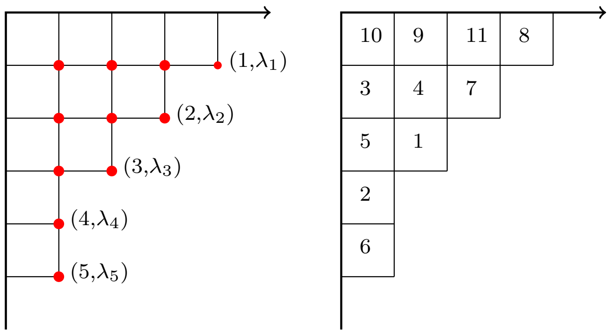
The cycles c_i are uniquely determined by σ, but as disjoint cycles commute, their order is arbitrary. The sequence of the lengths of the cycles c_i that appear in the cycle decomposition of σ, say $\lambda_1, \ldots, \lambda_r$, is called the *cycle type* of σ. We shall order the sequence decreasingly, that is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, and with that convension the sequence is uniquely defined. For example the permutation $(1,2)(3,4)(5,6,7)(9)(10)$ has cycle type $3, 2, 2, 1, 1$.

The formula that follows, which is easily verified (just follow what what it does to t_i), is sometimes usefull. We let τ be a permutation such that $\tau(i) = t_i$. Then

$$\tau(1, 2, 3, \ldots, r)\tau^{-1} = (t_1 \ldots, t_r)$$

The formula shows that any two cycles of the same length are conjugate, and therefore so are two permutations with the same cycle type. Consequently the conjugacy classes of S_n are in one-one correspondence with the set of cycle types. A cycle type is a decreasing sequence $\lambda_1, \lambda_2, \ldots, \lambda_r$ of natural numbers whose sum equals n (remeber that the cycles of length one also are included). This is what we call a *partion* of n.

The upshot of all this, is that the irreducible representations of S_n, being in a one-one correspondence with the conjugacy classes, are in a one-one correspondance with the partitions of n. It is very natural to ask for an explicit way of constructing an irreducible representation from a partition. There are several ways of doing this, and the one we shall follow is the old trail due to Alfred Young and Herman Weyl. In due course we shall give a formula for an *idempotent* $c_\lambda \in \mathbb{C}[S_n]$ depending on the partition λ, such that $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is irreducible. The idempotent c_λ is called a *Young symmetriser*.



Figur 1: To the left the diagram of the partition $4, 3, 2, 1, 1$. To the right a filling.

THE DIAGRAM OF A PARTITION. To any partion $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, one associates the *diagram* $D(\lambda)$ of λ. It is the subset of of $\mathbb{R}^2$ given by

$$D(\lambda) = \{ (i, \lambda_i) \mid 1 \leq i \leq r \}.$$

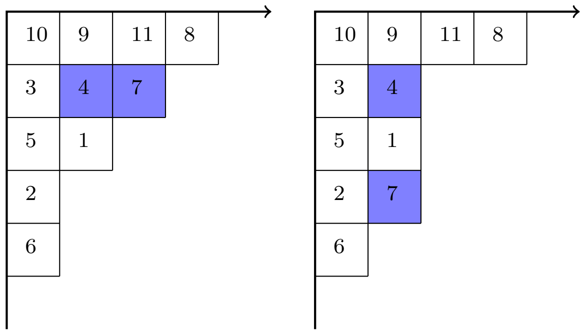
We usually draw it in a plane with the axes oriented as in a matrix, *i.e.,* the first coordinate increases down along the traditional y-axis, and the second increases to the rigth, along the traditional x-axis. The *boxes* of the diagram are the squares of unit sides filling out the area between the two axes and $D(\lambda)$, that is the unit squares having their lower right corner located at a diagram point.

The figure shows the diagram of the partion $4, 3, 2, 1, 1$ of 11. The points of $D(\lambda)$ are drawn as red dots.

YOUNG TABLEAUX The next step is to fill in the numbers between 1 and n in the boxes of the diagram, placing one number in each box. This is called a *filling* of the diagram, and the diagram with the numbers is often called a *Young tableau* or a *tablaeu* fro short. Formally, a filling is bijective map $T\colon D(\lambda) \to [1, n]$, which pictorially means that we place the number $T(i, j)$ in the box with lower right corner (i, j). The permutation group S_n acts on the fillings by permuting the numbers, formslly $\tau T = \tau \circ T$.

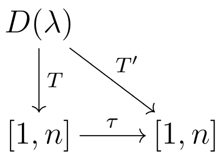
THE ROW GROUP AND THE COLUMN GROUP We are especially interested in two subgroups of S_n. The first one is the *row group* P of the filling. It consists of all permutations that respect the rows, *i.e.,* those that only permutes the numbers within each row. The other group is the *column group* Q. The members are the permutations respecting all the columns. Clearly $P \cap Q = \{e\}$.

For example the filling in the figure has a row group whose elements must fix 2 and 6, they can swap the numbers 5 and 1 and can permute $3, 4, 7$ as well as



Figur 2: The diagrams of two partitions λ and μ with $\lambda > \mu$. Two numbers in same row and same column in blue.

$8, 9, 10, 11$. The two groups P and Q depend on the filling. However if we change the filling by a permutation τ, the row group and the column group of $T' = \tau T$ will be the conjugates $\tau P \tau^{-1}$ and $\tau Q \tau^{-1}$ respectively. To see this, the relation between T and T' may be depicted with the commutative diagram:



and it should be clear that the numbers in the i-th row of $T' = \tau T$ is just the numbers one gets by applying τ to the numbers in the i-th row of T fro which it follows that P' and P are conjugate.

THE LEXICOGRAPHICAL ORDER The partions are partially ordere by *the lexicographical order*. Given two partitions λ and μ, then $\lambda \geq \mu$ if $\lambda_k > \mu_k$ where k is the first index where they differ. This is *total order*, fullfilling the usual axioms for a partial order, and given two different partitions, one of them is coming first. For example $4, 3, 1$ is ranked before $4, 2, 2$.

Assume that $\lambda > \mu$ and let T be a filling of λ and U one of μ. Then there are at least two numbers occuring in the same row of T and in the same column of U. Indeed, the number n being the last number in the first row of λ differing from the corresponding row of μ, must appear in one of the later rows of U. The corresponding column has a number m in the row where n occured in T. And then n and m are our numbers.

THE YOUNG SYMMETRISERS. Now we fix a partition λ and chose a filing T of λ. The corresponding row group and column group are denoted by P and Q respectively. We define two elements in the group algebra $\mathbb{C}[S_n]$:

$$a_\lambda = \sum_{p \in P} p$$

$$b_\lambda = \sum_{p \in Q} \operatorname{sign}(q)\, q.$$

And the most important gay, the *Young symmetriser* is the just their product

$$c_\lambda = a_\lambda b_\lambda.$$

Strictly speaking, this not a proper idempotent, there is a scalar involved, which however is easy to compensate for. We shall prove:

Theorem 1 *The symmetriser c_λ satisfies*

$$c_\lambda^2 = n_\lambda c_\lambda$$

where n_λ is an integer different from zero. Hence $n_\lambda^{-1} c_\lambda$ is an idempotent.

When we gave the definition of the Young symmetriser we made a choice of the filling T of the diagram. This dependence is however not to serious. Another filling T' gives another Young symmetriser b'_λ, but the two are conjugate by the permutation τ such that $\tau T = T'$, that is $\tau b_\lambda \tau^{-1} = b'_\lambda$. This follows easily since the two row groups and the two column groups are conjugate.

THE CLASSIFICATION OF THE IRREDUCIBLES. Then $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is an S_n module from the left. The points of the whole construction is the following theorem that we eventually shall prove:

Theorem 2 *The module V_λ is an complex, irreducible S_n-moduler. If $\lambda \neq \mu$, then the two modules V_λ and V_μ are not isomorphic.*

To make the notation simpler, for a while we'll drop the references to it by subscripts .

Lemma 1 *For any $p \in P$ and $q \in Q$ the following relations hold:*

$$ap = pa = a$$

$$bq = qb = \operatorname{sign}(q)\, b$$

$$pcq = \operatorname{sign}(q)\, c$$

The last one characterises c up to a scalar multiple.

PROOF: The two first ones are more or less obvious: A now standard argument gives $pa = \sum_{x \in P} px = a$ since px runs through the row group P when x does. Similarily, we have $qb = \sum_{x \in Q} q \operatorname{sign}(x)\, x = \operatorname{sign}(q) \sum_{q \in Q} \operatorname{sign}(qx)\, qx = \operatorname{sign}(q)\, b$ since, again, qx runs through Q when q does. The third equality, the one for c, follows immediatly from the two first. The only demanding thing to prove is that the last condition characterises c.

Assume to that end that $x \in \mathbb{C}[S_n]$ is an element satisfying $pxq = \operatorname{sign}(q)\, x$ for all $p \in P$ and $q \in Q$ and let $x = \sum_{y \in S_n} \alpha_y y$. Substituting the expression for x in the equality $pxq = \operatorname{sign}(q)\, x$, we get the following

$$p^{-1}xq^{-1} = \sum_{y \in S_n} \alpha_y p^{-1} y q^{-1} = \sum_z \alpha_{pzq} z = \sum_y \operatorname{sign}(q)\, \alpha_y y = \operatorname{sign}(q)\, x.$$

Equating coefficients we see that $\alpha_z = \alpha_{pzq}$ for all z. In particular $\alpha_{pq} = \alpha_e$, so if we can show that $\alpha_z = 0$ for all permutations z that are not in PQ we are safe.

Now we claim:

If g is a permutation not in PQ we may find to numbers n_1 and n_2 being in the same row in T and in the same column of $T' = g^{-1}T$.

Once this is established the proof is complete, indeed if τ is the transposition swapping n_1 and n_2, then $t \in P$ and $t \in Q'$, but $Q' = gQg^{-1}$, so $s = g^{-1}tg \in Q$. Then $tgs = tgg^{-1}tg = g$, and $\alpha_g = \alpha_{tgs} = \operatorname{sign}(s)\, \alpha_g = -\alpha_g$ since s is a transposition. Hence $\alpha_g = 0$ as we wanted.

We attack the claim by induction on the number n. We shall see that if two elements in a same row in $T' = g^{-1}T$ never are in the same column in T, then $g \in PQ$. Assuming this, no two elements in the first row of T' are in the same column of T, hence permuting within the columns of T, that is applying an element $q \in Q$, we can make the first rows of T' and qT identical. Then the rest of the two fillings are fillings of a smaller diagram, and since we changed T by an element in Q, the smaller fillings fullfill the hypothesis. By induction $qg^{-1} \in PQ$, and hence $g \in PQ$. ❑

The following gives a proof of the first part of the theorem:

Corollary 1 *For any element $x \in S$, the product cxc is scalar multiple of c.*

PROOF: This is a direct consequence of the lemma. Indeed, $pcxcq = pabxabq = abxab = cxc$, and by the third statement, the element cxc is a multiple of c. ❑

Theorem 3 *V_λ is irreducible.*

PROOF: Let $V = V_\lambda$ and $S = \mathbb{C}[S_n]$. Assume that $W \subseteq V$ is a nontrivial invariant subspace. Since $\mathbb{C}[S_n]$ is completely reducible, W is a direct summand of S. Stated differently, this means that there is an idempotent e such that $W = Se$.

Now $Wc \subseteq Vc \subseteq \mathbb{C}c$ by the corollary 1.

There are two possibilities. Either $cW = \mathbb{C}$ c or $cW = 0$.

In the latter case, $W \cdot W \subseteq VW = AcW = 0$. But $e \in W$, so it follows that $e = e^2 = 0$ and therefore $W = 0$. Hence the first possibility $cW = \mathbb{C}c$ occurs since W was supposed to be non-trivial. Then, however, $c = cae$ for some $a \in A$, and $V = Ac = Acae \subseteq Ae = W$, and we are done. ❑

Lemma 2 *Assume that $\lambda > \mu$ in the lexicographical oreder. Then $a_\lambda x b_\mu = 0$ for any $x \in S$. I particular $c_\lambda c_\mu = 0$.*

PROOF: By linearity we may assume that x is in S, *i.e.*, $x = \tau \in S$. Let T be the filling of diagram of λ used to construct a_λ and U the one of $D(\mu)$ used to construct b_μ.

We shall first show that $a_\lambda b_\mu = 0$, which is the salient point. If $\lambda > \mu$, we may, as in figure 3, find two numbers placed in the same row in the filling T and in the same column in U. Then the transposition σ swapping those two numbers gives $\alpha_\lambda \sigma = \alpha_\lambda$ and $\sigma b_\mu = -b_\mu$, which together implies $a_\lambda b_\mu = a_\lambda \sigma \sigma b_\mu = 0$.

Now if we use $U' = \tau U$ two construct a new b'_μ, we get $b'_\mu = \tau b_\mu \tau^{-1}$, and by what we saw, $0 = a_\lambda b'_\mu = a_\lambda \tau b_\mu \tau^{-1}$. Hence $a_\lambda \tau b_\mu = 0$. ❑

Theorem 4 *If λ and μ are two different partitions, then V_λ and $V\mu$ are not isomorphic*

PROOF: Now $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$ but $c_\lambda V_\mu = 0$ by lemma 2, so V_λ and V_μ are not isomorphic. ❑

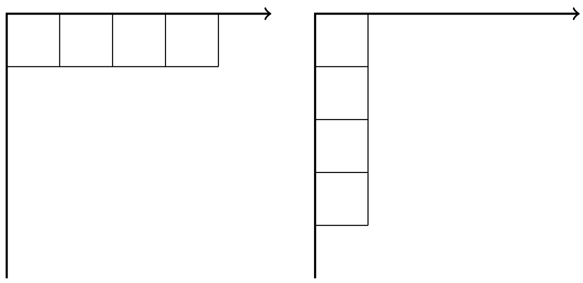
Tis finishes the proof of the main theorem in this section:

Theorem 5 *There is a one to one correspondence between partitions λ of the natural number n and irreducible complex, representations of the symmetric group S_n.It is given by associating to λ the module $V_\lambda = \mathbb{C}[S_n]c_\lambda$, where c_λ is the Young symmetriser.*

Finally, two examples:

EXAMPLE 1. — THE TRIVIAL REPRESENTATION. Take the stupid partion with just the one element n. Then the row group is the whole symmetric group S_n, and the column group is reduced to $\{1\}$. Then $c = \sum_{p \in S_n} p$, which is just the projection on to the fixed part of S_n. That is, the corresponding representation is the trivial one; indeed $gc = g$ for all $g \in S_n$. ❋

EXAMPLE 2. — THE ALTERNATION REPRESENTATION. The other extreme case is the partition $1, 1, 1, \ldots, 1$ consisting of n ones. Then the row group is reduced to the trivial group, and the column group is the whole symmetric group. The Young symmetriser is $c = \sum_{q \in S_n} \operatorname{sign}(q)\, q$, and the corresponding representation is the alternating one: We have $gc = \operatorname{sign}(g)\, c$ for any $g \in S_n$. ❋



Figur 3: The diagrams of the partitions giving the trivial and the alternating representation of S_4.