MAT4270: ADVANCED QUESTIONS

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These questions are about things slightly (or greatly) beyond what I can cover during the lecture, but I encourage you to think about them over the course. Some of these will be used as assignments for evaluation.

1. Group cohomology of finite groups

Notation. G: finite group, K: field of characteristic 0 (can be $\mathbb{C}$)

Why is $H^n(G; M)$ trivial for $n > 0$ if M is a representation of G over K?

- give a conceptual reasoning using $H^n(G; M) = \mathrm{Ext}^n_{K[G]}(K, M)$ and a categorical property of representations of G over K.
- give a concrete reasoning for $Z^n(G; M) = B^n(G; M)$ by "averaging" argument. Try $n = 1, 2$ first to get the hang of it, then general n if you have time.

References: [Bro94, Wei94]

2. Fixed point counting and positivity

Notation. G: finite group, X: finite G-set

For $g \in G$, let $f(g)$ be the number of fixed points,

$$f(g) = |\{x \in X \mid gx = x\}|\,.$$

Show that f is *positive definite*: that is, if $(g_i)_{i\in I}$ is a family of elements of G and $(c_i)_{i\in I}$ is a family of complex numbers on the same index set, we have

$$\sum_{i,j\in I} c_i \bar{c}_j f(g_j^{-1} g_i) \geq 0.$$

Hint: express $f(g)$ as the trace of some matrix, and use $\mathrm{Tr}(A^* A) \geq 0$ if A^* is the conjugate transpose of A.

Also: $f'(g) = |X| - f(g)$ is *conditionally negative definite*: if the coefficients $(c_i)_i$ satisfy $\sum_i c_i = 0$, then

$$\sum_{i,j\in I} c_i \bar{c}_j f'(g_j^{-1} g_i) \leq 0.$$

This implies that $\exp(-f'(g))$ is positive definite.

Reference: [BO08]

3. Hopf algebras

Notation. G: finite group

What are the Hopf algebras associated with G, and how are they related?

- explain the Hopf algebra structure on the group algebra $\mathbb{C}[G]$.
- do the same for the function algebra $\mathcal{O}(G)$.

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- explain the duality between $\mathbb{C}[G]$ and $\mathcal{O}(G)$.
- formulate the notion of representation of G in terms of $\mathcal{O}(G)$.

Reference: [Car07]

4. Deformation problem

Notation. $\mathfrak{g}$: Lie algebra (say over $\mathbb{R}$) with bracket $[x, y]$

Suppose that we are given an infinitesimal deformation of $\mathfrak{g}$. Concretely, that means we are given another map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]'$ such that $[x, y] + \epsilon[x, y]'$ defines a Lie algebra over $\mathbb{R}[\epsilon]/(\epsilon^2)$ when interpreted as the natural ϵ-linear extension on $\mathfrak{g}[\epsilon]/(\epsilon^2) = \{x_1 + \epsilon x_2 \mid x_1, x_2 \in \mathfrak{g}\}$. Here ϵ is an 'infinitesimal' such that $\epsilon^2 = 0$.

- what is the consistency condition between $[x, y]$ and $[x, y]'$?
- describe the above relation in terms of Lie algebra cohomology; interpret $[x, y]'$ as a cochain in the Chevalley–Eilenberg cochain complex of $\mathfrak{g}$ with coefficient $\mathfrak{g}$ itself. Which $[x, y]'$ correspond to the coboundaries?
- present the Lie algebra of $ax + b$ group as a deformation of the commutative Lie algebra $\mathbb{R}^2$.
- what does Whitehead's lemma imply for semisimple Lie algebras?

References: [Kna88, Wei94]

5. Flag manifold

Notation. $\mathbb{P}^1(\mathbb{C})$: complex projective space ($\mathbb{C} \cup \{\infty\}$, topologically S^2)

Explain geometric realization of the irreducible representations of SL_2.

- find a representation of $\mathfrak{sl}_2(\mathbb{C})$ as algebraic vector fields on $\mathbb{P}^1(\mathbb{C})$.
- what are the holomorphic line bundles on $\mathbb{P}^1(\mathbb{C})$ (usually denoted $\mathcal{O}(n)$ for $n \in \mathbb{Z}$), and what are the space of holomorphic sections $H^0(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n)) = \Gamma(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$?
- describe the first cohomology $H^1(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$, either through holomorphic differential forms in these coefficients, or through the Čech cohomology for the decomposition $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup_{\mathbb{C}^\times} \mathbb{C}$.
- identify these with the spaces of homogeneous polynomials in two variables, and compare the actions of $\mathfrak{sl}_2$ (or SL_2).
- interpret this construction as the Borel–Weil–Bott construction.

Reference: [Sep07]

6. Coadjoint action

Notation. $\mathfrak{g}$: Lie algebra, $\mathfrak{g}^*$: the linear dual of $\mathfrak{g}$, $\mathrm{Sym}(\mathfrak{g})$: the symmetric algebra

$$\mathrm{Sym}(\mathfrak{g}) = \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \right) / \langle x \otimes y - y \otimes x \mid x, y \in \mathfrak{g} \rangle .$$

What is the Kirillov bracket on $\mathfrak{g}^*$?

- interpret the elements of the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$ as functions on $\mathfrak{g}^*$.
- extend the bracket $[x, y]$ on $\mathfrak{g}$ to a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$.
- explain the relation between the universal enveloping algebras of $\mathfrak{g}$ with rescaled brackets the deformation quantization of this Poisson bracket.

7. Loop groups and affine Kac–Moody algebras

Notation. G: simple compact Lie group

What is a loop group LG, and what is its Lie algebra?

- how do you realize these objects in the following settings? (and what are the merits?)
  - rational • smooth • continuous
- what is the affine Lie algebra $\hat{\mathfrak{g}}$, and what is the affine Kac–Moody algebra $\tilde{\mathfrak{g}}$? (the terminology varies among different authors)
- describe its Cartan matrix and Dynkin diagram of $\tilde{\mathfrak{g}}$.

Reference: [Kac90]

8. Deformation quantization

Notation. M: C^∞-manifold

Explain the Lie algebraic concepts in the problem of Poisson deformation quantization.

- what is a polyvector field on M? Explain the graded Lie algebra structure (Schouten–Nijenhuis bracket) on the space of polyvector fields $\Gamma(\bigwedge^* TM)$.
- what is a polydifferential operator on M? Explain the differential graded Lie algebra structure on the space of polydifferential operators $D^*(M)$.
- how are they related? What are the Maurer–Cartan elements in these differential graded Lie algebras?
- explain the Poisson deformation quantization problem in terms of the above objects.

9. Drinfeld–Kohno Lie algebra

Notation. $\mathfrak{t}_n$: the nth Drinfeld–Kohno Lie algebra, G: simple compact Lie group

- describe the generators and relations of $\mathfrak{t}_n$.
- what is the relation between $\mathfrak{t}_n$ and the pure braid group PB_n?
- how are they related to the configuraton space of n distinct complex numbers?
- when (π, V) is a representation of G, explain that there is an induced representation of $\mathfrak{t}_n$ on $V^{\otimes n}$.

References

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