§1 Linear representation of finite groups.

Main ref. Serre "Linear repr. of fin. groups.".

Blanket assumption : work over fin. dim. cplx vec. sps. ($\simeq \mathbb{C}^n$ for some $n = 0, 1, \cdots$)

Notn $\mathbb{C} = \{ a + b\sqrt{-1} : a, b \text{ real} \}$

V : f.d. ($\mathbb{C}$-)vec sp ($\simeq \mathbb{C}^n$)

$\mathrm{End}(V) = \{ T : V \to V \text{ lin.} \} \simeq M_n(\mathbb{C})$

$GL(V) = \{ T \in \mathrm{End}(V) \text{ invertible} \} \simeq GL_n(\mathbb{C})$

$= \{ X \in M_n(\mathbb{C}) : \det X \neq 0 \}$

Id_V : identity map $\longleftrightarrow I_n$.

G : finite group.

linear rep. of G :

concrete) system of matrices $(X_g)_{g \in G}$

$\underset{\substack{\uparrow \\ \text{fixed}}}{X_g \in M_n(\mathbb{C})}$, $X_g X_h = X_{gh}$, $X_{\underset{\substack{\uparrow \\ \text{neutral elem.}}}{e}} = I_n$

abstract) V : f.d. vec. sp.

$\pi : G \to GL(V)$ group hom.

Convention : π_g instead of $\pi(g)$

so $\pi_g v \in V$ when $v \in V$.

Sometimes gv instead of $\pi_g v$

so $g(hv) = (gh)v$, $ev = v$.

First examples

1. $G = S_n = \{ \sigma : \{1, \cdots, n\} \to \{1, \cdots, n\} \text{ bijective} \}$

$\{\sigma(1), \cdots, \sigma(n)\} = \{1, \cdots, n\}$.

$V = \langle e_i : i = 1, \cdots, n \rangle$. ($\simeq \mathbb{C}^n$)

$= \{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{C} \}$.

$\pi_\sigma e_i = e_{\sigma(i)}$ $\quad (\pi_\sigma \pi_\tau e_i = e_{\sigma\tau(i)} = \pi_{\sigma\tau} e_i)$

2. $G = \mathbb{Z}/n\mathbb{Z} = \{ [i] : i \in \mathbb{Z},\ [i] = [i+kn] \}$

1-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$

$$\varphi^{(1)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} i} \in \mathbb{C} = M_1(\mathbb{C})$$

well-defied

$$\varphi^{(1)}_{[i]} \varphi^{(1)}_{[j]} = e^{\frac{2\pi\sqrt{-1}}{n}(i+j)} = \varphi^{(1)}_{i+j}.$$

More generally : $\varphi^{(k)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} k i}$

$\varphi^{(0)}, \cdots, \varphi^{(n-1)}$ are the only 1-dim reps

$\because$ $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times = GL_1(\mathbb{C})$ multiplicative group

- determined by $\varphi_{[1]}$ $\quad (\varphi_{[i]} = \varphi_{[1]}^{\,i})$
- $\varphi_{[1]}^{\,n} = \underbrace{\varphi_{[1]} \cdots \varphi_{[1]}}_{n\times} = \varphi_{n[1]} = \varphi_{[0]} = 1$

$\Rightarrow$ $\varphi_{[1]}$ should be n-th root of unity.

Questions

- How do we compare different reps?
- Decomposition of reps
  $\rightsquigarrow$ indecomposable / irreducible reps
- Classification by characters $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$.