Today's summary

- Direct sum of reps
- Comparison of reps

) → Complete reducibility, irreducible reps

- Group ring

(- Tensor product)

Notn : G finite group

(π, V) linear rep of G ($\pi : G \to GL(V)$).

("V: "underlying space" of π)

Direct sum of reps.

(π, V), (π', V') rep. of G ⇝ "$\pi \oplus \pi'$" on $V \oplus V'$

$V \oplus V' = \{ v \oplus w : v \in V, w \in V' \} \simeq V \times V'$

$(\pi \oplus \pi')_g (v \oplus w) = (\pi_g v) \oplus (\pi'_g w)$

Concrete) $(X_g)_{g \in G}$ $(X'_g)_{g \in G}$ $X_g \in M_m(\mathbb{C})$, $X'_g \in M_n(\mathbb{C})$

⇝ $\begin{bmatrix} X_g & 0 \\ 0 & X'_g \end{bmatrix} \in M_{m+n}(\mathbb{C})$

Comparison of reps.

<u>Intertwiner</u> (or (G-)<u>homomorphism</u>) from (π, V) to (π', V')

$T : V \to V'$ linear map s.t. $\forall g \in G$ $T \pi_g = \pi'_g T$

Concrete) with above $(X_g)_g$, $(X'_g)_g$: $T \in M_{n \times m}(\mathbb{C})$

$\forall g$ $T X_g = X'_g T$

Notn : $\mathrm{Hom}_G(V, V') = \{ T : V \to V' \text{ intertwiner} \}$

$\mathrm{Hom}_G((\pi, V), (\pi', V'))$, $\mathrm{Hom}(\pi, \pi')$, (π, π'), ...

π and π' are <u>isomorphic</u> (or <u>equivalent</u>, <u>similar</u>)

if $\exists$ <u>bijective</u> intertwiner $T : V \to V'$.

Example $G = \mathbb{Z}/n\mathbb{Z}$

- $V = \langle e_{[i]} : [i] \in G \rangle$ $\quad \pi_{[i]} e_{[j]} = e_{[i+j]}$

  (restriction of $S_n \curvearrowright V$ from yesterday
  $G \ni [1] \leftrightarrow$ cycl. perm $(1\ 2 \cdots n) \in S_n$)

- $\psi^{(0)} \oplus \cdots \oplus \psi^{(n-1)}$ on $\mathbb{C}^n$

  $\psi^{(k)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} ik}$, $\quad f_k = 0 \oplus \cdots \oplus \underset{k\text{-th}}{1} \oplus \cdots \oplus 0$

$$T : V \to \mathbb{C}^n, \quad e_{[i]} \mapsto \sum_{k=0}^{n-1} e^{\frac{2\pi\sqrt{-1}}{n} ik} f_k$$
$$= 1 \oplus e^{\frac{2\pi\sqrt{-1}}{n} i} \oplus \cdots \oplus e^{\frac{2\pi\sqrt{-1}}{n} i(n-1)}$$

Exercise : check $(\psi^{(0)} \oplus \cdots \oplus \psi^{(n-1)})_{[1]} T e_{[i]} = T \pi_{[1]} e_{[i]}$

What did we do ? :

$\pi_{[1]} \leftrightarrow$ "rotation" matrix $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ & 1 & \ddots & \vdots \\ & & \ddots\ 0 & \\ & & 1 & 0 \end{bmatrix}$

relation between $T e_{[i]}$ and f_k

$\leftrightarrow$ rel. between $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i$ & eigenvector

Complete reducibility

(π, V) rep of G. $W \subset V$ subspace is G-invariant if $\forall w \in W$, $g \in G$ $\pi_g w \in W$.

$\rightsquigarrow$ $((\pi_g|_W)_g, W)$ is again rep of G.

subrep of (π, V)

Th'm. Under above setting $\exists W' \subset V$ s.t.

- W' is complement to W : $V = W \oplus W'$
- W' is G-invariant.

Rec! (Hermitian) inner product (v, v') $v, v' \in V$

$(v, v') \in \mathbb{C}$, $\|v\|^2 = (v, v) \geq 0$, $\|v\| = 0 \Leftrightarrow v = 0$

$(v, v') = \overline{(v', v)}$, linear in v

Proof. Step 1 V has a G-invar. inner product.

$$(v, v') = (\pi_g v, \pi_g v')$$

$\because$ $(v, v')_0$: any inn. prod (from $V \simeq \mathbb{C}^n$ say)

$$(v, v') = \sum_{h \in G} (\pi_h v, \pi_h v')_0$$

$$\rightsquigarrow (\pi_g v, \pi_g v') = \sum_h (\pi_h \pi_g v, \pi_h \pi_g v')_0$$

$$= \sum_h (\pi_{\underbrace{hg}} v, \pi_{\underbrace{hg}} v')_0 = (v, v')$$

relabel as h'

Step 2 $W' = \{ w' \in V : \forall w \in W \quad (w, w') = 0 \}$

orthogonal complement.

is G-invar.

$$\because (w, \pi_g w') \underset{\text{Step 1}}{=} (\pi_{g^{-1}} w, \pi_{g^{-1}} \pi_g w) = (\underbrace{\pi_{g^{-1}} w}_{\in W}, w')$$

$$= 0 .$$

$\boxtimes$

Rem. Step 1 $\Rightarrow$ any rep of fin. grp. is <u>unitarizable</u>

unitary rep : $\pi : G \to U(H)$

H : Hilbert sp. (Herm. inn. prod. sp.)

$U(H)$: unitary group.

$= \{ T : H \to H, \ (T\xi, T\eta) = (\xi, \eta) \}$

(π, V) is <u>irreducible</u> if only subreps are the obvious ones $W = 0, V$

$0 \neq V$

Cor. Any rep is isom. to direct sum of irreducible ones

$\because$ V "irred" $\Rightarrow$ "direct sum with one summand"

$V = 0 \quad \Rightarrow$ "zero summand"

not irred $\Rightarrow$ take $W \subset V \underset{\text{Thm}}{\Rightarrow} V \simeq W \oplus W'$

subreps

keep decomposing W & W'.

Tensor product.

(π, V), (π', V') reps. $\rightsquigarrow$ <u>tensor product</u> rep.

$\pi \otimes \pi'$ on $V \otimes V'$.

$$V \otimes V' = \Big\{ \sum_{\alpha_i} \alpha_i v_i \otimes v'_i \quad v_i \in V,\ v'_i \in V',\ \alpha_i \in \mathbb{C} \Big\}$$

$$(v_1 + v_2) \otimes v' = v_1 \otimes v' + v_2 \otimes v'$$
$$(\alpha v) \otimes v' = \alpha (v \otimes v') = v \otimes \alpha v' \quad \text{etc.}$$

$$(\pi \otimes \pi')_g (v \otimes v') = \pi_g v \otimes \pi'_g v'$$

Group algebra

$$\mathbb{C}[G] = \Big\{ \sum_{g \in G} \alpha_g \cdot g \;:\; \alpha_g \in \mathbb{C} \Big\}$$

$$\Big(\sum_g \alpha_g g\Big)\Big(\sum_h \beta_h h\Big) = \sum \alpha_g \beta_h \underbrace{(gh)}_{\text{computed in } G}$$
$$= \sum_k \Big(\sum_h \alpha_{kh^{-1}} \beta_h\Big) k.$$

Representations of G $\equiv$ modules over $\mathbb{C}[G]$

irred. reps $\equiv$ simple modules

Complete reducibility $\Leftrightarrow$ any $\mathbb{C}[G]$-module is a direct sum of simple ones

$\mathbb{C}[G]$ is <u>semisimple</u>

$$\underset{\substack{\text{general theory}\\ \text{of semisimple algs}}}{\Longrightarrow} \quad \mathbb{C}[G] \simeq \prod_{(\pi, V):\,\text{irred rep.}} \mathrm{End}(V)$$