Summary

- Characters
- Schur's lemma (intertwiners between irred. reps)
- Orthogonality of irred. chars.

G: finite group.

(π, V), (π', V'), … : linear reps of G.

<u>Character</u> of (π, V): $\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) = \mathrm{Tr}_{\mathrm{End}(V)}(\pi(g))$

Rem

- $\chi_\pi(g) =$ sum of eigenvalues of $\pi(g)$

  $\pi(g)$ can be represented by a unitary mat.

  $\Rightarrow \chi_\pi(g) \in \mathbb{Z}[e^{\frac{2\pi\sqrt{-1}}{|G|}}]$

- $\mathrm{Tr}_{\mathrm{End}(V)}(ABA^{-1}) = \mathrm{Tr}_{\mathrm{End}(V)}(B)$

  $\Rightarrow \chi_\pi(ghg^{-1}) = \chi_\pi(h)$ $(g, h \in G)$.

  χ_π is a <u>class function</u> (const. on each conjugacy class).

Prop. $\chi_{\pi \oplus \pi'}(g) = \chi_\pi(g) + \chi_{\pi'}(g)$, $\chi_{\pi \otimes \pi'}(g) = \chi_\pi(g)\chi_{\pi'}(g)$

$\because$ $(e_i)_{i=1}^m$ basis of V, $(f_j)_{j=1}^n$ basis of V'

$\Rightarrow$ $(e_1 \oplus 0, \ldots, e_m \oplus 0, 0 \oplus f_1, \ldots, 0 \oplus f_n)$: basis of $V \oplus V'$

$(e_1 \otimes f_1, \ldots, e_m \otimes f_1, e_1 \otimes f_2, \ldots, e_m \otimes f_n)$ basis of $V \otimes V'$

$\Rightarrow$ $\mathrm{Tr}_{\mathrm{End}(V \oplus V')}(A \oplus B) = \mathrm{Tr}_{\mathrm{End}(V)}(A) + \mathrm{Tr}_{\mathrm{End}(V')}(B)$

$\mathrm{Tr}_{\mathrm{End}(V \otimes V')}(A \otimes B) = \mathrm{Tr}_{\mathrm{End}(V)}(A)\, \mathrm{Tr}_{\mathrm{End}(V')}(B)$

for $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(V')$

Use $A = \pi(g)$, $B = \pi'(g)$ $\square$

Example $\overset{!}{\nearrow}$ sometimes "character" means this !

- 1-dim rep $\phi : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

  $\chi_\phi(g) = \phi(g)$

- $\pi : S_n \curvearrowright V = \langle e_1, \cdots, e_n \rangle$

  $\chi_\pi(\sigma) = \mathrm{Tr}_{M_n(\mathbb{C})}$ (permutation matrix for σ)

  = number of 1's in the diagonal

  $= \# \{ i : \sigma(i) = i \}$ (number of fixed points)

Schur's lemma) $(\pi, V), (\pi', V')$ irreducible reps of G. $T : V \to V'$ intertwiner.

1) π & π' are not isomorphic : T must be 0 (write $\pi \not\cong \pi'$)

2) π & π' isomorphic : if $T \neq 0$, T is invertible & any other intertwiner is a scalar multiple of T.

Proof : Basic idea : $\mathrm{Im}\, T$, $\mathrm{Ker}\, T$ are G-invariant subspaces

$\because$ $\pi'(g) T v = T \pi(g) v \in \mathrm{Im}\, T$

$v \in \mathrm{Ker}\, T \Rightarrow T \pi(g) v = \pi'(g) T v = 0$

$\Rightarrow \pi(g) v \in \mathrm{Ker}\, T$.

Proof of 1). (proof by contradiction)

if $\left.\begin{array}{l} T \neq 0 \quad \mathrm{Ker}\, T \neq 0 \\ \text{irreducibility of } \pi \end{array}\right) \Rightarrow \mathrm{Ker}\, T = 0$

$\mathrm{Im}\, V \neq 0$ & irred'ty of π' $\Rightarrow \mathrm{Im}\, T = V'$

$\Rightarrow T$ is bijective. π & π' isom.

Proof of 2) By the same arg. $T \neq 0 \Rightarrow T$ invertible

$S : V \to V'$ another intertwiner

λ : any eigenvalue of $T^{-1} S \in \mathrm{End}(V)$)

- $T^{-1}S - \lambda \mathrm{Id}_V$ is an intertwiner from π to π.
- $T^{-1}S - \lambda \mathrm{Id}_V$ has nontrivial kernel

$\Rightarrow$ $T^{-1}S - \lambda \mathrm{Id}_V = 0$ i.e. $S = \lambda T$.

Averaging maps into intertwiners

Prop. (π, V), (π', V') reps.

$T : V \to V'$ linear map.

$\Rightarrow$ $\tilde{T} = \frac{1}{|G|} \sum_{h \in G} \pi'(h) \, T \, \pi(h)^{-1}$ is an intertwiner.

Proof Step 1 $\tilde{T}$ is intertwiner $\Leftrightarrow$ $\forall g \in G$ $\pi'(g) \tilde{T} \pi(g)^{-1} = \tilde{T}$

Step 2 $\pi'(g) \tilde{T} \pi(g)^{-1} = \frac{1}{|G|} \sum_h \pi(g)\pi(h) \, T \, \pi(h)^{-1} \pi(g)^{-1}$

$= \frac{1}{|G|} \sum_h \underbrace{\pi(gh) \, T \, \pi(gh)^{-1}}_{\text{relabel as } h'} = \tilde{T}$

Cor. π, π' irred., $T : V \to V'$ linear, $\tilde{T}$ as above.

1) $\pi \not\cong \pi'$ $\Rightarrow$ $\tilde{T} = 0$
2) $V = V'$, $\pi = \pi'$ $\Rightarrow$ $\tilde{T} = \frac{\mathrm{Tr}(T)}{\dim V} \mathrm{Id}_V$

$\because$ for 2) $\cdot$ $\mathrm{Tr}(\tilde{T}) = \frac{1}{|G|} \sum_h \underbrace{\mathrm{Tr}\left(\pi(h) \, T \, \pi(h)^{-1}\right)}_{= \mathrm{Tr}(T)} = \mathrm{Tr}(T)$

$\cdot$ $\tilde{T}$ is a scalar mult. of Id_V.

Orthogonality of irred. chars.

$\varphi, \psi : G \to \mathbb{C}$ maps $\Rightarrow$ Hermitian inner product

$$(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

Th'm 1) π irred. rep $\Leftrightarrow$ $(\chi_\pi, \chi_\pi)_{L^2(G)} = 1$.

2) π': another irred. rep, $\pi \not\cong \pi'$ $\Rightarrow$ $(\chi_\pi, \chi_{\pi'})_{L^2(G)} = 0$

Before proof,

Examples 1) $\varphi, \psi : G \to GL_1(\mathbb{C})$ 1-dim reps

$\varphi \simeq \psi$ as reps $\Leftrightarrow$ $\varphi = \psi$ as maps.

$(\varphi, \varphi')_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \underbrace{|\varphi(g)|^2}_{\text{all } 1} = 1$.

$(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_g \varphi(g) \overline{\psi(g)} = 0$ if $\varphi \neq \psi$.

2) $S_n \curvearrowright V = \langle e_1, \cdots, e_n \rangle$.

$v_0 = e_1 + \cdots + e_n$ is fixed by any $\sigma \in S_n$

$\pi|_{\mathbb{C}v_0}$ is isom. to the trivial rep
$(\varphi(\sigma) = 1)$

W : S_n-invariant complement of $\mathbb{C}v_0$

$\Rightarrow$ $V \simeq \mathbb{C}v_0 \oplus W$ as reps. $\pi' := \pi|_W$.

$\underbrace{\chi_\pi(\sigma)}_{\#\{i : \sigma(i) = i\}} = \underbrace{1}_{\text{char. of triv. rep}} + \chi_{\pi'}(\sigma')$

$(\chi_{\pi'}, \chi_{\pi'}) = \frac{1}{n!} \sum_{\sigma \in S_n} \left(\#\{i : \sigma(i) = i\} - 1\right)^2$

Fact : Right hand side is always 1.
($\Rightarrow$ π' is irreducible)

Check for $n = 3$

σ	e	$(1,2), (2,3), (3,1)$	$(1,2,3), (3,2,1)$
$\chi_\pi(\sigma)$	3	1	0
$\chi_{\pi'}(\sigma)$	2	0	-1.