Summary

- Contragredient & adjoint reps
- Proof of orthogonality
- Structure of characters
  - Regular representation

G : fin. grp. (π, V), (π', V') : fin. reps. of G

Rem $\pi \simeq \pi' \Rightarrow \chi_\pi = \chi_{\pi'}$ (we'll later see $\Leftarrow$).

$\because$ $T : V \to V'$ bijective intertwiner

$\mathrm{Tr}_{\mathrm{End}(V')}(TAT^{-1}) = \mathrm{Tr}_{\mathrm{End}(V)}(A)$ for $A \in \mathrm{End}(V)$

$A = \pi_g \Rightarrow TAT^{-1} = \pi'_g$

<u>Contragredient</u> <u>rep</u> :

$V^* = \{ \xi : V \to \mathbb{C} \text{ linear} \}$ dual space of V

$\rightsquigarrow$ rep π^c of G on V^* by $(\pi^c_g \xi)(v) = \xi(\pi_{g^{-1}} v)$

$$(\pi^c_g(\pi^c_h \xi))(v) = (\pi^c_h \xi)(\pi_{g^{-1}} v) = \xi(\pi_{h^{-1}} \pi_{g^{-1}} v)$$

$$= \xi(\pi_{(gh)^{-1}} v) = (\pi^c_{gh} \xi)(v)$$

Concretely, $(X_g)_{g \in G}$ matrices in $M_n(\mathbb{C})$

corresponding to π $\rightsquigarrow$ $X^c_g = (X_{g^{-1}})^t$ corresp. to π^c.

If X_g is unitary, X^c_g is componentwise conjugate

$\overline{X_g} := (\overline{(X_g)_{ij}})_{i,j}$.

<u>Adjoint</u> <u>rep</u> : $\mathrm{Hom}(V, V') = \{ T : V \to V' \text{ linear} \}$

(also write $\mathrm{Hom}_{\mathbb{C}}(V, V')$, $\mathrm{Lin}(V, V')$, $B(V, V')$, $\cdots$)

$\mathrm{Ad}_g(T) = \pi'_g T \pi_{g^{-1}}$ defs rep. of G on $\mathrm{Hom}(V, V')$

Rem $V' \otimes V^* \simeq \mathrm{Hom}(V, V')$ by $\sum v'_i \otimes \xi_i \mapsto (v \mapsto \sum \xi_i(v) v'_i)$

this is a intertwiner from $\pi' \otimes \pi^c$ to Ad.

Translating Schur's lemma

Suppose $\sum v_i' \otimes \xi_i \in V' \otimes V^*$ corr. to $T \in \mathrm{Hom}(V, V')$

T is an intertwiner $\Leftrightarrow T = \pi_g' T \pi_g^{-1} \quad \forall g$

$\Leftrightarrow \sum v_i' \otimes \xi_i$ is fixed by $\pi_g' \otimes \pi_g^c$.

When π, π' irred

1) $\pi \not\simeq \pi'$ $\nexists$ G-invar. nonzero vec. in $V' \otimes V^*$

2) $\pi \simeq \pi'$ $\exists$ up to scalar unique G-inv. vec. in $V' \otimes V^*$.

Proof of orthogonality (almost)

Step 1) $(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \frac{1}{|G|} \sum_g \chi_{\pi'}(g) \overline{\chi_\pi(g)}$

is the trace of $\frac{1}{|G|} \sum_g \pi_g' \otimes \pi_g^c$

$\because \mathrm{Tr}_{\mathrm{End}(V' \otimes V^*)}(\pi_g' \otimes \pi_g^c) = \mathrm{Tr}_{\mathrm{End}(V')}(\pi_g') \, \mathrm{Tr}_{\mathrm{End}(V^*)}(\pi_g^c)$

So it's enough to have $\overline{\chi_\pi}(g) = \mathrm{Tr}_{\mathrm{End}(V^*)}(\pi_g^c)$

- $(X_g)_g$ corr to $\pi \Leftrightarrow ((X_{g^{-1}})^t)_g$ corr. to π^c

$\Rightarrow \mathrm{Tr}((X_{g^{-1}})^t) = \mathrm{Tr}(X_{g^{-1}}) = \mathrm{Tr}((X_g)^{-1})$

- λ : eigenval. of $X_g \Rightarrow |\lambda| = 1$.

$\Rightarrow \mathrm{Tr}((X_g)^{-1}) = \sum_{\substack{\lambda : \text{e-val} \\ \text{of } X_g}} \lambda^{-1} = \sum_\lambda \bar{\lambda} = \overline{\mathrm{Tr}(X_g)}$

- Conceptually : using G-inv. inn. prod

$(\pi^c, V^*) \simeq (\bar{\pi}, \bar{V})$ $\quad \xi(v) = (v, w)$ for $\exists ! w$

$\bar{V} = \{ \bar{v} : v \in V, \ \lambda \bar{v} = \overline{\lambda v} \}$ conjug. sp.

$\bar{\pi}_g \bar{v} = \overline{\pi_g v} \qquad \mathrm{Tr}(\bar{\pi}_g) = \overline{\mathrm{Tr}(\pi_g)}$

Step 2) Set $\Phi = \frac{1}{|G|} \sum_g \pi_g' \otimes \pi_g^c$.

$\mathrm{Im}\, \Phi$ = G-invariant vectors in $V' \otimes V^*$

$\because (\pi_h' \otimes \pi_h^c) \Phi = \frac{1}{|G|} \sum_g \pi'_{\underbrace{hg}_{\text{relabel as } g'}} \otimes \pi^c_{hg} = \Phi$.

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Step 3) Proof of claim (2)

π, π' irred., $\pi \not\simeq \pi' \Rightarrow \Phi = 0$

$\because$ By Schur's lem. $\not\exists$ G-inv. vec. in $V' \otimes V^*$.

$\Rightarrow \mathrm{Im}\,\Phi = 0$

Step 4) Proof of "$\Rightarrow$" for claim (1)

π irred $\Rightarrow \mathrm{Tr}\,\Phi = 1$.

$\because$ • $\mathrm{Im}\,\Phi$ is 1-dim (span of $\sum_i e_i \otimes e^i$ corr. to Id_V ; $(e_i)_i$ basis of V, $(e^i)_i$ dual basis)

• $\Phi^2 = \Phi$ from $(\pi'_h \otimes \pi^c_h)\Phi = \Phi$ ∴ average of left hand side is Φ^2 証

Structure of characters.

(π, V) rep of G. Suppose

$$(\pi, V) \simeq \underbrace{(\pi_1, V_1) \oplus \cdots \oplus (\pi_1, V_1)}_{n_1 \times} \oplus \underbrace{(\pi_2, V_2) \cdots \oplus (\pi_2, V_2)}_{n_2 \times} \oplus \cdots \oplus \underbrace{(\pi_k, V_k) \oplus \cdots \oplus (\pi_k, V_k)}_{n_k \times}$$

• each (π_i, V_i) irred.

• $i \neq j \Rightarrow \pi_i \not\simeq \pi_j$.

$$= \bigoplus_{i=1}^{k} (\pi_i, V_i)^{\oplus n_i}$$

Then $\chi_\pi = n_1 \chi_{\pi_1} + n_2 \chi_{\pi_2} + \cdots + n_k \chi_{\pi_k}$ and

• $(\chi_\pi, \chi_\pi) = n_1^2 + n_2^2 + \cdots + n_k^2$

• $n_i = (\chi_\pi, \chi_{\pi_i})$: "multiplicity of π_i in π

$\because (\chi_{\pi_i}, \chi_{\pi_j}) = \delta_{ij}$

So • $(\chi_\pi, \chi_\pi) = 1 \Rightarrow \pi$ is irred.

("$\Leftarrow$" for claim (1) of orthogonality)

• irred decomp of π is unique up to rearranging direct summands

(and replacing summands by isom. irreps.)

<u>Regular representation</u> (λ, V) with

$V = \langle e_g : g \in G \rangle$. $\lambda_g e_h = e_{gh}$

(other form : underlying vec. sp $\mathbb{C}(G) = L^2(G)$

$(\lambda_g f)(h) = f(g^{-1} h)$

$V \ni e_g \leftrightarrow \delta_g \in L^2(G)$)

Concretely) $\lambda_g \leftrightarrow (x_{h,k})_{h,k \in G}$ with

$x_{h,k} = \begin{cases} 1 & h = gk \\ 0 & h \neq gk \end{cases}$

$\Rightarrow \chi_\lambda(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$

Prop. $(\pi_1, V_1), \dots, (\pi_k, V_k)$ all irred. reps of G (up to isom.)

$\pi_i \not\simeq \pi_j$ for $i \neq j$

Then $(\lambda, V) \simeq \bigoplus_{i=1}^{k} (\pi_i, V_i)^{\oplus \dim V_i}$

$\because$ mult. of π_i in λ $= (\chi_{\pi_i}, \chi_\lambda)$

$= \frac{1}{|G|} \sum_g \chi_{\pi_i}(g) \chi_\lambda(g) = \chi_{\pi_i}(e) = \mathrm{Tr}(\underbrace{\pi_i(e)}_{Id_{V_i}})$

$= \dim V_i$

Examples 1) $G = \mathbb{Z}/n\mathbb{Z}$. $V = \langle e_{[i]} : [i] \in \mathbb{Z}/n\mathbb{Z} \rangle$

$(\lambda, V) \simeq \bigoplus_{k=0}^{n-1} \varphi^{(k)}$ for $\varphi^{(k)}([i]) = e^{\frac{2\pi\sqrt{-1}}{n} k i}$

2) $G = S_3$. Irreducible reps :

two 1-dim : $\pi^{triv}(\sigma) = 1$. $\pi^{sig}(\sigma) = (-1)^{|\sigma|}$.

one 2-dim : π' : complement of π^{triv}.

for $S_3 \curvearrowright \langle e_1, e_2, e_3 \rangle$

$\underset{\text{6-dim}}{\lambda} \simeq \pi^{triv} \oplus \pi^{sig} \oplus \pi' \oplus \pi'$

Rem λ corresponds to $\mathbb{C}[G] \curvearrowright \mathbb{C}[G]$ left. mult.

We have $\mathbb{C}[G] \simeq \prod_{\pi_i : \text{irrep.}} \mathrm{End}(V_i) \simeq \prod_i V_i \otimes V_i^*$

left mult. becomes $\pi_i \otimes 1$ on $V_i \otimes \underset{\text{gives mult.}}{V_i^*}$