Summary
- Counting irreducible representations ($= \#(\text{conjug. cls})$)
- Character table

- Counting irred. reps.

G : finite group (π, V) lin. rep. of G

$f : G \to \mathbb{C}$ map $\rightsquigarrow$ $\pi_f \in \mathrm{End}(V)$ by

$$\pi_f(v) = \sum_{g \in G} f(g)\, \pi_g v .$$

Rem. this corresponds to $\mathbb{C}[G]$-module structure on V

f is a <u>class function</u> if $f(hgh^{-1}) = f(g)$ $\forall g, h \in G$

Ex. $\chi_{\pi'}$ is a class func.

Prop. 1) f class func. $\Rightarrow$ $\pi_f \in \underset{\substack{\uparrow \\ \text{intertwiners from } \pi \text{ to } \pi}}{\mathrm{End}_G(V)}$

2) f class func. & π irred

$\Rightarrow$ π_f is the scalar mult. by $\dfrac{|G|}{\dim V} (f, \overline{\chi_\pi})_{L^2(G)}$

$$= \frac{1}{\dim V} \sum_{g \in G} f(g)\, \chi_\pi(g) .$$

Proof 1) Want : $\pi_{h^{-1}} \pi_f \pi_h = \pi_f$

$$\pi_{h^{-1}} \pi_f \pi_h = \sum_{g \in G} f(g)\, \pi_{h^{-1}} \pi_g \pi_h = \sum_g f(g)\, \pi_{\underbrace{h^{-1} g h}_{\text{relabel as } g'}}$$

$$= \sum_{g'} f(hg'h^{-1})\, \pi_{g'} \underset{f \text{ is a class func.}}{=} \pi_f .$$

2) π_f is scalar (say α) by 1) & Schur's lemma.

$$\alpha = \frac{1}{\dim V} \mathrm{Tr}_{\mathrm{End}(V)}(\pi_f) = \frac{1}{\dim} \sum_g f(g)\, \underbrace{\mathrm{Tr}_{\mathrm{End}(V)}(\pi_g)}_{\chi_\pi(g)} .$$

$\square$

$Z(\mathbb{C}[G]) : \{f : G \to \mathbb{C} \text{ class func.}\}$.

Th'm The irreducible characters of G form an orthonormal basis of $Z(\mathbb{C}[G])$ (w.r.t. $(-,-)_{L^2(G)}$)

Proof. $\pi^1, \cdots, \pi^k$ all irred. reps. of G, $\chi_i = \chi_{\pi^i}$
$V_1 \quad V_k$ (up to isom, mutually nonisom)

We alredy know that $(\chi_i)_i$ is orthonormal

$\Rightarrow$ Enough to show $Z(\mathbb{C}[G]) \underset{\text{orth. compl.}}{\ominus} \langle \chi_1, \cdots, \chi_k \rangle = 0$

i.e. $f \in Z(\mathbb{C}[G])$, $\forall i$ $(f, \chi_i)_{L^2(G)} = 0 \Rightarrow f = 0$

Step 1. f as above $\Rightarrow \pi^i_f = 0$

$\because$ $(\pi^i)^c$ is also irred. & $\chi_{(\pi^i)^c}(g) = \overline{\chi_i(g)}$

$\Rightarrow \overline{\chi_i} = \chi_{\bar{i}}$ for some $\bar{i}$ $\underset{\text{assumption.}}{\Rightarrow} (f, \overline{\chi}_i) = 0$

$\underset{\text{Prop. 2)}}{\Rightarrow} \pi^i_f = 0$

Step 2. $\pi^i_f = 0$ for $\forall i \Rightarrow \lambda_f = 0$ (λ : reg. rep.)

$\because$ $\lambda \simeq \bigoplus_{i=1}^{k} (\pi^i)^{\oplus \dim V_i}$ (see last time)

$\Rightarrow \lambda_f \simeq \bigoplus (\pi^i_f)^{\oplus \dim V_i} = 0$

Step 3. $\lambda_f = 0 \iff f = 0$

$\because$ $V = \langle \delta_g : g \in G \rangle$ underlying sp. of λ

$\lambda_f \delta_e = \sum_{g \in G} f(g) \lambda_g \delta_e = \sum_g f(g) \delta_g$

So $\lambda_f = 0 \Rightarrow \lambda_f \delta_e = 0 \Rightarrow f = 0$ ☑

Abstract version)

f is a class function $\iff \sum_{g \in G} f(g) \cdot g$ is in the <u>center</u> of $\mathbb{C}[G]$.

(same computation as Prop 1))

We also know $\mathbb{C}[G] \cong \prod_{i=1}^{k} \mathrm{End}(V_i)$.

Center of $\mathrm{End}(V_i) = \mathbb{C}\,\mathrm{Id}_{V_i}$

$\Rightarrow \underset{\text{center}}{Z}\left(\prod_{i=1}^{k} \mathrm{End}(V_i)\right) \simeq \mathbb{C}^k \Rightarrow \dim Z(\mathbb{C}[G]) = k.$

Rem. $\dim Z(\mathbb{C}[G]) = \#(\text{conjugacy classes of } G)$

$\because$ $C_1, \cdots, C_{k'}$ conj. cls. $\Rightarrow (\delta_{C_i})_{i=1}^{k'}$ basis of $Z(\mathbb{C}[G])$

Cor. $g \in G$ 1) $c(g) = |\{hgh^{-1} : h \in G\}|$

$\sum_{i=1}^{k} |\chi_i(g)|^2 = \frac{|G|}{c(g)}$

2) h not conjug. to $g \Rightarrow \sum_{i=1}^{k} \chi_i(h)\overline{\chi_i(g)} = 0$

Proof. Step 1. f class func $\Rightarrow f = \sum_{i=1}^{k} \alpha_i \chi_i$

$\alpha_i = (f, \chi_i)_{L^2(G)}$

$\because$ $(\chi_i)_i$ is ONB.

Step 2. $X = \{hgh^{-1} : h \in G\}$.

$\Rightarrow \delta_X = \sum_{i=1}^{k} \frac{c(g)}{|G|} \overline{\chi_i(g)}\, \chi_i$

$\because$ $(\delta_X, \chi_i) = \frac{1}{|G|} \sum_{g' \underset{\text{conj}}{\sim} g} \overline{\chi_i(g')}$ [δ_X is a clss. func.

Step 3 1) : $1 = \delta_X(g) \underset{\text{Step 2}}{=} \frac{c(g)}{|G|} \sum_i \overline{\chi_i(g)}\chi_i(g)$

2) : $h \not\sim g \Rightarrow 0 = \delta_X(h) = \frac{c(g)}{|G|} \sum_i \overline{\chi_i(g)}\chi_i(h)$

Character table.

Draw table with

- column : conjugacy classes (or representatives g)
- row : irreducible characters χ_i
- entries : $\chi_i(g)$

- Cyclic groups : $G = \mathbb{Z}/n\mathbb{Z}$

	$[0]$	$[1]$	$\cdots$	$[n-1]$
$\chi^{(0)}$	1	1	$\cdots$	1
$\chi^{(1)}$	1	$e^{\frac{2\pi\sqrt{-1}}{n}}$	$\cdots$	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\chi^{(n-1)}$	1	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)}$	$\cdots$	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)^2}$

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- $G = S_3$: irred. chars : χ_{triv}, χ_{sig}, $\chi_{\pi'}$ (last time)

conjug. classes $\{e\}$, $\{(1\,2), (2\,3), (3\,1)\}$, $\{(1\,2\,3), (3\,2\,1)\}$

	e	$(1\,2)$	$(1\,2\,3)$	
χ_{triv}	1	1	1	
χ_{sig}	1	-1	1	
$\chi_{\pi'}$	2	0	-1	($=\#$(fixed points) -1)

Ex. $c((1\,2)) = 3$ $\quad |\chi_{triv}((1\,2))|^2 + |\chi_{sig}((1\,2))|^2 + |\chi_{\pi'}((1\,2))|^2$

$$= 1 + 1 = \frac{6}{3} = \frac{|S_3|}{|c((1\,2))|}$$

$$\chi_{triv}((1\,2))\,\chi_{triv}((1\,2\,3)) + \chi_{sig}((1\,2))\,\chi_{sig}((1\,2\,3)) + \chi_{\pi'}((1\,2))\,\chi_{\pi'}((1\,2\,3))$$

$$= 1\cdot 1 + (-1)\times 1 + 0 \times 1 = 0 .$$

- $G = A_4 = \{\sigma \in S_4 : (-1)^{|\sigma|} = 1\}$

($= \{T \in SO(3) : T\Delta = \Delta\}$ $\quad \Delta$: reg. tetrahedron)

$|A_4| = \frac{|S_4|}{2} = 12$. $\qquad A_4 = K_4 \rtimes (\mathbb{Z}/3\mathbb{Z})$

conjug. classes. $\{e\}$. $\{(i\,j)(k\,l) : i \sim l$ all diff$\}$ three elems.

$\{(1\,2\,3), (2\,4\,3), (2\,1\,4), (3\,4\,1)\}$

$\{(3\,2\,1), (3\,4\,2), (4\,1\,2), (1\,4\,3)\}$

irreps : three 1-dim (from $\mathbb{Z}/3\mathbb{Z}$)

one 3-dim. χ_π

	e	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(3\,2\,1)$
$\mu^{(0)}$	1	1	1	1
$\mu^{(1)}$	1	1	$e^{\frac{2\pi\sqrt{-1}}{3}}$	$e^{\frac{4\pi\sqrt{-1}}{3}}$
$\mu^{(2)}$	1	1	$e^{\frac{4\pi\sqrt{-1}}{3}}$	$e^{\frac{2\pi\sqrt{-1}}{3}}$
χ_π	3	-1	0	0

π : from $A_4 \to SO(3) \curvearrowright \mathbb{R}^3 \underset{sc.\,ext}{\rightsquigarrow} \mathbb{C}^3$