Summary
- Frobenius reciprocity
  - duality
- McKay correspondence
  (classification of finite subgroups of $SU(2)$)

Frobenius reciprocity

$G$ finite group, $(\pi, V)$, $(\pi', V')$, $(\pi'', V'')$ : reps of $G$

$$\dim \mathrm{Hom}_G(V, V' \otimes V'') = \dim \mathrm{Hom}_G(V \otimes \underbrace{(V'')^*}_{\text{contragred. rep}}, V')$$

Rem. $\pi$ irred, $(\rho, W)$ another rep

$\Rightarrow \dim \mathrm{Hom}_G(V, W)$ is "the number of times" $\pi$ happens in $\rho$.

$\rho = \pi_1 \oplus \cdots \oplus \pi_k$ irr. decomp.

$$\Rightarrow \dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(V, \oplus V_i)$$
$$= \sum \underbrace{\dim \mathrm{Hom}_G(V, V_i)}_{1 \text{ if } \pi \simeq \pi_i,\ 0 \text{ otherwise}}$$

Proof of reciprocity.

Step 1 : $(e_i)_i$ basis of $V$, $(e^i)_i$ dual basis of $V^*$

$\Rightarrow R_\pi : \mathbb{C} \to V^* \otimes V$, $\alpha \mapsto \alpha(\sum_i e^i \otimes e_i)$

$\bar{R}_\pi : \mathbb{C} \to V \otimes V^*$, $\alpha \mapsto \alpha(\sum_i e_i \otimes e^i)$

are intertwiners.

$\because \sum e_i \otimes e^i \leftrightarrow Id_V \in \mathrm{End}(V)$
fixed by $\mathrm{Ad}_g(T) = \pi_g T \pi_g^{-1}$

Step 2 With (invariant) inner prod. on $V$,

$$(\bar{R}_\pi^* \otimes Id_V)(Id_V \otimes R_\pi) = Id_V$$

$(T : V \to V' \rightsquigarrow T^* : V' \to V$, $(Tv, v') = (v, T^* v'))$

$\because (e_i)_i$ : orthonormal basis $\Rightarrow R(\alpha) = \sum_i \bar{e}_i \otimes e_i$, etc.

$$(\bar{R}^* \otimes Id)(Id \otimes R) = (\sum e_i^* \otimes \bar{e}_i^* \otimes Id)(\sum Id \otimes \bar{e}_j \otimes e_j)$$
$$= \sum_i e_i^* \otimes e_i = Id.$$

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Step 3 $\mathrm{Hom}_G(V, V'\otimes V'') \to \mathrm{Hom}_G(V\otimes(V'')^*, V')$,

$$T \mapsto (\mathrm{Id}_{V'}\otimes \bar{R}^*_{\pi''})(T\otimes \mathrm{Id}_{(V'')^*})$$

$$\mathrm{Hom}_G(V\otimes(V'')^*, V') \to \mathrm{Hom}_G(V, V'\otimes V'')$$

$$S \mapsto (S\otimes \mathrm{Id}_{V''})(\mathrm{Id}_V\otimes R_{\pi''})$$

are inverse to each other.

$\because$ Want: $(\mathrm{Id}\otimes \bar{R}^*_{\pi''}\otimes \mathrm{Id})(T\otimes \mathrm{Id}_{(V'')^*\otimes V''})(\mathrm{Id}_V\otimes R_{\pi''}) = T.$

By $(T\otimes \mathrm{Id}_{(V'')^*\otimes V''})(\mathrm{Id}_V\otimes R_{\pi''}) = T\otimes R_{\pi''}$
$= (\mathrm{Id}\otimes R_{\pi''})T$

we can reduce this to step 2. $\square$

McKay correspondence

Finite subgroups of $SU(2)$ $\leftrightarrow$ finite groups with 2-dim faithful representation by "det = 1" matrices.

$SU(2) = \{U \in M_2\mathbb{C} : U^*U = I_2,\ \det U = 1\}$

$SO(3) = \{X \in M_3\mathbb{R} : X^t X = I_3,\ \det X = 1\}$.

Prop. $\exists\ \varphi : SU(2) \to SO(3)$, $\ker\varphi = \{\pm I_2\}$.

Proof. $V_0 = \{X \in M_2\mathbb{C} : X^* = -X,\ \mathrm{Tr}\,X = 0\}$

3-dim real sp. with basis $\begin{bmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{bmatrix}, \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$

inner prod $(X, Y) = \mathrm{Tr}(X^*Y) = -\mathrm{Tr}(XY)$.

(above basis = $2\times$ orthonormal basis)

$SU(2)$ acts by $\mathrm{Ad}_U(X) = UXU^*$

preserving $(X, Y)$ $\rightsquigarrow$ $SU(2) \to SO(3)$ $\square$

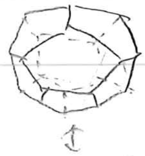
$G \subset SU(2)$ finite subgroup $\Rightarrow \varphi(G) \subset SO(3)$

Possibilities of $\varphi(G)$:

- cyclic groups $\mathbb{Z}/n\mathbb{Z}$ : rotation around one axis
- dihedral groups $(\mathbb{Z}/n\mathbb{Z})\rtimes_{\pm 1}(\mathbb{Z}/2\mathbb{Z})$ : symmetry of regular polygon.
- symmetries of regular polyhedra

$G(3,3,2)$

$G(4,3,2)$



$G(5,3,2)$

20 triangles

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Possibilities of $G$:

- cyclic groups $\mathbb{Z}/n\mathbb{Z}$ : $\left\langle \begin{bmatrix} e^{\frac{2\pi\sqrt{-1}}{n}} & 0 \\ 0 & e^{-\frac{2\pi\sqrt{-1}}{n}} \end{bmatrix} \right\rangle$
- inv. img of dihedr. grps $(\mathbb{Z}/2n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
- inv. img of reg. polyhed. grps.

Drawing graph $\Gamma = (V, E)$ out of $G$

- vertex set $V = \{\pi_0, \cdots, \pi_k$ : irred. reps of $G\}$
  $\pi_0$ : trivial rep.
- edges $\pi_i \to \pi_j$ : as many as $A_{ij} = \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho)$

Prop. 1) $\Gamma$ is unoriented $A_{ij} = A_{ji}$

2) $A_{ii} = 0$ unless $G$ is trivial

3) $A_{ij} \leq 1$ $(i \neq j)$

Proof 1) $\rho$ is self dual $(\rho, \mathbb{C}^2) \simeq (\rho^c, (\mathbb{C}^2)^*)$

isom : $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$

i.e. $R : \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2$, $\alpha \mapsto \alpha(e_1 \otimes e_2 - e_2 \otimes e_1)$

$\bar{R} = -R$ give a model of $(\rho^c, \mathbb{C}^{2*})$

$A_{ji} = \dim \mathrm{Hom}_G(\pi_j, \pi_i \otimes \rho) \underset{\text{Frob.}}{=} \dim \mathrm{Hom}_G(\pi_j \otimes \rho^c, \pi_i)$

$\underset{\rho^c \cong \rho}{=} \dim \mathrm{Hom}_G(\pi_j \otimes \rho, \pi_i) = \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho)$

$T \mapsto T^*$

2) $A_{ii} \leq 2$ by dim counting $(\dim(\pi_i \otimes \rho) = 2 \dim \pi_i)$

$A_{ii} = 1$ cannot happen.

$\because f : \mathrm{Hom}_G(\pi_i, \pi_i \otimes \rho) \to \mathrm{Hom}_G(\pi_i, \pi_i \otimes \rho)$

$T \mapsto (T^* \otimes Id)(Id \otimes R)$

is conjug. lin & $f^2 = -Id$.

But conj. lin $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is $\alpha \mapsto \bar{\alpha}\lambda$ $(\lambda = f(1))$

$\rightsquigarrow f^2(\alpha) = \alpha|\lambda|^2 \neq -\alpha$.

$A_{ii} = 2$ means $\pi_i \otimes \rho \cong \pi_i \oplus \pi_i$.

$\exists k : \lambda \subset \rho^{\otimes k} \Rightarrow \pi_i^{\oplus 2^k} \supset \lambda$. $\Rightarrow$ $G$ trivial.

3) $\sum_{j=0}^{k} A_{ij}^2 = (\chi_{\rho\otimes\pi_i}, \chi_{\rho\otimes\pi_i})_{L^2(G)} = \frac{1}{|G|}\sum_g |\chi_\rho(g)|^2 |\chi_{\pi_i}(g)|^2$

$|\chi_\rho(g)| \le 2$ and $\frac{1}{|G|}\sum_g |\chi_{\pi_i}(g)|^2 = 1$

$\Rightarrow \sum_j A_{ij}^2 \le 4$.

Inequality happens $\Rightarrow$ $|A_{ij}| \le 1$

Equality happens $\Rightarrow$ $\forall g$ $\underbrace{|\chi_\rho(g)| = 2}_{\text{only for } g = \pm I_2}$ or $|\chi_{\pi_i}(g)| = 0$

$A_{ij} = 2 \Rightarrow \rho\otimes\rho_i \simeq \rho_j \oplus \rho_j$, $\rho\otimes\rho_j \simeq \rho_i \oplus \rho_i$

$-I_2$ acts trivially on $\rho_i$ or $\rho_j$ (say $\rho_i$)

$\rightsquigarrow$ $\rho_i$ is an irred. rep. of $H = G/G\cap\{\pm I_2\}$.

$\rightsquigarrow$ $\chi_{\rho_i}(h) = d_i \delta_e(h)$ on $H$ ($d_i = \dim V_i$)

i.e. reg. character of $H$ belongs to the lin. span of $\chi_{\rho_i}$ $\Rightarrow$ $H$ trivial. $\Rightarrow$ $\rho_i$ also

$\Rightarrow$ $\rho$ trivial. $\Rightarrow$ $i = j$, $G$ triv. ☒

Prop. $d_i = \dim V_i$

1) $2d_i = \sum A_{ij} d_j$

2) $\chi_\rho(g)\chi_{\pi_i}(g) = \sum A_{ij}\chi_{\pi_j}(g)$

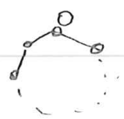
Proof. 1) $2d_i = \dim(\rho\otimes\pi_i) = \dim \bigoplus \pi_j^{\oplus A_{ji}}$

$= \sum \underbrace{A_{ji}}_{\overset{\shortparallel}{A_{ij}}} d_j$

2) use same decomp, take character.

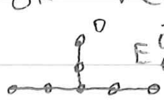
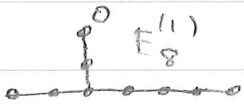
Recap: $\Gamma$ has norm 2 $\|A\|_{B(\ell^2 V)} = 2$

without loops around vertices (…)

$G = \mathbb{Z}/n\mathbb{Z}$ $\rightsquigarrow$  $A_n^{(1)}$ $n+1$ vertices.

inv. img. of $(\mathbb{Z}/2(k-2)) \rtimes (\mathbb{Z}/2\mathbb{Z})$  $D_k^{(1)}$ $(k+1)$-vertices

inv. img of reg. polyhedra grps

 $E_6^{(1)}$  $E_7^{(1)}$  $E_8^{(1)}$