Summary
- Lie algebras
  - subalgs, ideal, center
- Representation of Lie algs
- Universal enveloping algs

∘ Lie algebra

K : field ($\mathbb{R}$ or $\mathbb{C}$ for us)

A Lie alg. over K is :

- K-vector space $\mathfrak{g}$ (fin. dim for us)
- bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$ s.t. "bracket"

$[X, Y] = -[Y, X]$ (alternating)

$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

(Jacobi identity)

Rem : Lie alg $\equiv$ infinitesimal model of continuous grps (tomorrow)

Examples

1) $[X, Y] = 0$ for all X, Y. (commutative)

2) V = K-vector space, $\mathfrak{g} = \mathrm{End}(V)$ (as lin. sp.)
bracket $[X, Y]v = XYv - YXv$. $(v \in V)$
to emphasize this bracket we write $\mathfrak{gl}(V) = \mathfrak{g}$.
$\mathfrak{gl}_n(K) = \mathfrak{gl}(K^n)$ $(= M_n(K)$ as lin. sp.)

2') $\mathfrak{sl}(V) = \{ X \in \mathfrak{gl}(V) : \mathrm{Tr}(X) = 0 \}$
$\mathfrak{sl}_n(K) = \mathfrak{sl}(K^n)$.

3) $\mathfrak{u}_n = \{ X \in \mathfrak{gl}_n(\mathbb{C}) : X^* (= \bar{X}^t) = -X \}$ real Lie algs
$\mathfrak{su}_n = \{ X \in \mathfrak{u}_n : \mathrm{Tr}\, X = 0 \} = \{ X \in \mathfrak{sl}_n(\mathbb{C}) : X^* = -X \}$

Rem. $\mathfrak{g}$ real Lie alg. $\rightsquigarrow$ $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \{ \sum_{i=1}^{k} X_i \otimes \alpha_i : X_i \in \mathfrak{g}, \alpha_i \in \mathbb{C} \}$
cplx Lie alg.

Ex. $(\mathfrak{su}_n)_{\mathbb{C}} \simeq \mathfrak{sl}_n(\mathbb{C})$, $(\mathfrak{u}_n)_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$.
$\mathfrak{su}_2$ has basis $\begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}$, $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, $\begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$.
(skew-Hermitian) Pauli matrices

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these are basis of $\mathfrak{sl}_2(\mathbb{C})$: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = -\frac{1}{2}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \sqrt{-1}\begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}\right)$

<u>homomorphism</u> of Lie algs : $f : \mathfrak{g}_1 \to \mathfrak{g}_2$

linear map s.t. $[f(X), f(Y)] = f([X,Y])$.

<u>Lie subalgebra</u> of $\mathfrak{g}$: subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t.

$X, Y \in \mathfrak{h} \Rightarrow [X, Y] \in \mathfrak{h}$.

<u>ideal</u> of $\mathfrak{g}$: $\mathfrak{h} \subset \mathfrak{g}$ subspace, $X \in \mathfrak{g}$, $Y \in \mathfrak{h} \Rightarrow [X, Y] \in \mathfrak{h}$

Examples.

1) $\mathfrak{sl}_n(\mathbb{K}) \underset{\text{ideal}}{\subset} \mathfrak{gl}_n(\mathbb{K}) \underset{\text{ideal}}{\supset} \mathbb{K} I_n$ ($Tr([X, Y]) = 0$, $[X, I_n] = 0$.

$\Rightarrow \mathfrak{gl}_n(\mathbb{K}) = \mathfrak{sl}_n(\mathbb{K}) \oplus \mathbb{K}$

2) $\mathfrak{c} = \mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ [X, Y] = 0 \}$

<u>center</u> of $\mathfrak{g}$: ideal.

$\mathfrak{g}$ is <u>simple</u> if 0 and $\mathfrak{g}$ are the only ideals.

- $\mathfrak{sl}_n \mathbb{K}$, $\mathfrak{su}_n$ simple

3) $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} : a, b \in \mathbb{K} . \right\}$. $\left[\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \right] = \begin{bmatrix} 0 & 2(ab'-ba') \\ 0 & 0 \end{bmatrix}$.

$\mathfrak{h}_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in \mathbb{K} \right\}$ subalg but not ideal ($\mathrm{Char}\,\mathbb{K} \neq 2$)

$\mathfrak{h}_2 = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{K} \right\}$ ideal

4) $\mathfrak{so}_n(\mathbb{K})$

- Representation of Lie algebras

$\mathfrak{g}$: Lie alg over $\mathbb{K}$, V : vec. sp. over $\mathbb{K}$ write $\pi : \mathfrak{g} \curvearrowright V$

Linear <u>reprecention</u> of $\mathfrak{g}$ over V is given by :

linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ $X \mapsto \pi_X$ s.t.

$\pi_X \pi_Y v - \pi_Y \pi_X v = \pi_{[X,Y]} v$ $(v \in V)$

i.e. Lie alg hom $\mathfrak{g} \to \mathfrak{gl}(V)$

Rem. (π, V) rep of $\mathfrak{g}$ $\leftrightarrow$ Lie alg. structure on $\mathfrak{g} \oplus V$

$\supset \mathfrak{g}$ subalg

$\supset V$ ideal s.t.

- $[v, v'] = 0$

$[X \oplus v, X' \oplus v'] = [X, X'] \oplus (\pi_X v' - \pi_{X'} v)$

Example : adjoint representation $\mathrm{ad}: V = \mathfrak{g}$, $\mathrm{ad}_X Y = [X, Y]$

Jacobi identity $\Leftrightarrow [\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}$.

center $\mathfrak{z}(\mathfrak{g}) \ni X \Leftrightarrow \mathrm{ad}_X = 0$.

Fact : (Ado's theorem) $\dim_K \mathfrak{g} < \infty \Rightarrow \exists$ faithful rep. of $\mathfrak{g}$. ($X \neq 0 \Rightarrow \pi_X \neq 0$) (difficult)

Unitary representation : $\mathfrak{g}$: real Lie alg.

V : $\mathbb{C}$-vec. sp. with Hermitian inn. prod. ($\subset$ Hilbert sp.)

$\pi : \mathfrak{g} \curvearrowright V$ with $(\pi_X v, v') + (v, \pi_X v') = 0$

Ex. $\mathfrak{g} = \mathbb{R}$ (with zero bracket),

$V = \mathcal{S}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C}$ rapid decay $\forall m,n \ \sup_x |x^m \partial_x^n f(x)| < \infty \}$

$\pi_t f = t \partial_x f$ $\qquad (f_1, f_2) = \int f_1(x) \overline{f_2(x)} dx$

- Universal enveloping algebra

$\mathfrak{g}$: Lie alg. over K. $\mathcal{U}(\mathfrak{g})$: (unital) K-alg. with

- generators : $X \in \mathfrak{g}$
- relations : $\underbrace{X \cdot Y - Y \cdot X}_{\text{computation in } \mathcal{U}(\mathfrak{g})} = \underbrace{[X, Y]}_{\text{computation in } \mathfrak{g}}$

Formally : tensor algebra $T(\mathfrak{g}) = K \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots$

with prod. $(X_1 \otimes \cdots \otimes X_m) \cdot (Y_1 \otimes \cdots \otimes Y_n) = X_1 \otimes \cdots \otimes X_m \otimes Y_1 \otimes \cdots \otimes Y_n$

take bilateral ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$

$\rightsquigarrow \mathcal{U}(\mathfrak{g})$: quotient.

Rem. A K-alg. $\rightsquigarrow \mathfrak{gl}(A)$: A as vec sp. $[a, b] = ab - ba$.

Lie alg hom $\mathfrak{g} \to \mathfrak{gl}(A) \equiv$ alg hom $\mathcal{U}(\mathfrak{g}) \to A$.

i.e. left adjoint functor of $A \mapsto \mathfrak{gl}(A)$.

Rem. representation of $\mathfrak{g}$ = $\mathcal{U}(\mathfrak{g})$-module

$\pi : \mathfrak{g} \curvearrowright V \rightsquigarrow \pi : \mathcal{U}(\mathfrak{g}) \curvearrowright V$ by

$\pi_{X_1 \cdots X_m} v = \pi_{X_1} \cdots \pi_{X_m} v$

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Poincaré – Birkhoff – Witt theorem) $(X_i)_{i=1}^N$ basis of $\mathfrak{g}$ (N can be infinite) $\Rightarrow$ $\mathcal{U}(\mathfrak{g})$ has a basis $(X_1^{i_1} \cdots X_k^{i_k})_{k<\infty,\, k\le N,\, i_j \ge 0}$

Idea: any "noncomm. monomial" $X_{m_1} X_{m_2} \cdots X_{m_n}$ is a linear combination of above by induction on Σm_i. E.g.:

$$X_3 X_1 = X_1 X_3 + \underbrace{[X_3, X_1]}_{\text{can be written as sum of } (X_i)_i}$$

Rem. $F_n = \langle Y_1 \cdots Y_m : m \le n,\, Y_i \in \mathfrak{g} \rangle_{K\text{-span}}$ increasing seq. of subspaces, $F_n F_{n'} \subset F_{n+n'}$

$\mathcal{U}(\mathfrak{g}) = \bigcup_{n=0}^{\infty} F_n \Rightarrow \mathcal{U}(\mathfrak{g})$ is a <u>filtered</u> alg

PBW $\Rightarrow$ assoc. graded $\bigoplus_{n=0}^{\infty} (F_n / F_{n-1}) \simeq \underbrace{\mathrm{Sym}(\mathfrak{g})}_{\text{symmetric alg}}$.

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4) $\mathfrak{so}_n(K) = \{ X \in M_n(K) : X^t = -X,\ \mathrm{Tr}\, X = 0 \}$

$[X, Y]^t = (XY - YX)^t = [Y^t, X^t] = [Y, X] = -[X, Y]$

$\rightsquigarrow$ Lie subalg of $\mathfrak{sl}_n(K)$

Rem $\mathfrak{su}_2 \simeq \mathfrak{so}_3(\mathbb{R})$

$\mathfrak{so}_3(\mathbb{R})$ has basis $Y_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

compare with Pauli matrices & figure out rescaling factor