Summary
- Topological groups
  - Lie groups
- Lie groups to Lie algebras

∘ Topological groups.

Hausdorff top. sp. with group structure

$G \times G \to G$ $(g, h) \to gh$, $G \to G$ $g \mapsto g^{-1}$ cont.

Ex. discrete groups (discr. top.)

$\mathbb{R}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, ...

$\mathbb{Z}_p = \lim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z} = \{ \sum_{k=0}^{\infty} a_k p^k : 0 \leq a_k < p \}$

<u>Lie group</u> is a top. grp. s.t.
- $\exists$ neighborhood U of e, homeo. to an open set of $\mathbb{R}^n$ ($\varphi : U \hookrightarrow \mathbb{R}^n$ coord. func.)
- group laws are smooth around e

$U \times U \to G$, $(g, h) \mapsto gh^{-1}$ "looks like" a smooth map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ around (e, e)

Alternatively: top. group which is a C^∞-mfd.

(C^1-mfd is enough).

$G = \bigcup_{i\in I} U_i$ $\varphi_i : U_i \hookrightarrow \mathbb{R}^n$ img open

$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ smooth

& smooth group law.

Examples
- $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $SO_n(\mathbb{R})$, ... $\mathbb{R}^n$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, ...
- $U_n \subset GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$

$\uparrow \mathbb{C}^n \simeq \mathbb{R}^{2n}$ as real vec. sp.

∘ Lie groups to Lie algs

Tangent space at g = { "tangent vectors" at g }

$\rightsquigarrow$ formalize as "directional derivatives"

$g \in G$, U neigh. $U \xrightarrow{\varphi} \mathbb{R}^n$ homeo to open set

f : smooth func. around g $\rightsquigarrow$ $f \circ \varphi^{-1}$: smooth. around $\varphi(g) =: p$

$\rightsquigarrow$ partial derivatives $\partial_{x_i} f \circ \varphi^{-1}(p)$

$\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ $\rightsquigarrow$ $\xi_g(f) = \sum \alpha_i \partial_{x_i} f \circ \varphi^{-1}(p)$

such ξ_g's span an n-dim real vec. sp.

Intrinsically : $T_g G = \{ \xi : C^\infty(G) \to \mathbb{R}$
$\xi(f_1 f_2) = f_1(g)\xi(f_2) + \xi(f_1) f_2(p) \}$

is the <u>tangent space</u> at g. <u>Leibniz rule</u> at g

· invariant vector fields

G Lie group. $\xi \in T_e G$. $\rightsquigarrow$ two ways to get "translate" of ξ at $g \in G$:

ξ^l_g : $f \mapsto ((R_g)_\# \xi)(f) = \boxed{\xi(f \circ R_g)}$ $\quad R_g : h \mapsto hg$

ξ^r_g : $f \mapsto ((L_g)_\# \xi)(f) = \boxed{\xi(f \circ L_g)}$ $\quad L_g : h \mapsto gh$

satisfy the Leibniz rule at g

$f \in C^\infty(G) \rightsquigarrow (\xi^l_g(f))_{g \in G}$ is smooth in g.

$\xi^l = (\xi^l_g)$ is a <u>vector field</u> (the <u>right invariant</u> vec. field corresp. to ξ)

$\xi^l_{gh} = (R_g)_\# \xi^l_h$. right invariance

similarly $\xi^r = (\xi^r_g)_g$ <u>left invariant</u> vec. field

Prop. $G = GL_n(\mathbb{R})$ $f_{ij}(g) = g_{ij}$ (i,j)-entry of g

$F = (f_{ij})_{i,j=1}^n$ matrix of smooth funcs.

$X \in M_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$

0) $X(f) = \partial_t f(e^{tX})|_{t=0}$ $\quad (f \in C^\infty(G))$

is a tangent vec. at $e = I_n$.

1) $X(f_{ij}) = X_{ij}$

2) $(X^l(f_{ij}))_{i,j} = X \cdot F$ matrix prod.

3) $(X^r(f_{ij}))_{i,j} = F \cdot X$

Proof. 1) $X(f_{ij}) = \partial_t$ ((i,j)-comp. of $I_n + tX + \frac{t^2}{2}X^2 + \cdots$)

$= (i,j)$-comp. of $\partial_t (I_n + tX + O(t^2))|_{t=0}$

$= X_{ij}$

2) $X^\ell(f_{ij})(g) = X(f_{ij} \circ R_g)$

$f_{ij} \circ R_g : h \mapsto (i,j)$-comp. of hg

$= \sum_k f_{ik}(h) g_{kj}$

$= \sum_{k=1}^n f_{ik}\, g_{kj}$

So $X(f_{ij} \circ R_g) = \sum X(f_{ik}) g_{kj} = \sum X_{ik} g_{kj}$.

$\rightsquigarrow$ as a func. in g, eq. to XF.

3) similar □

Lie alg structure on $\mathfrak{g} = T_e G$

$X, Y \in T_e G \rightsquigarrow X^\ell, Y^\ell$ right inv. vec. fields

$\Rightarrow [X^\ell, Y^\ell](f) = X^\ell(\underbrace{Y^\ell(f)}_{g \mapsto Y^\ell_g(f)}) - Y^\ell(X^\ell(f))$ "comm. bracket"

is still a right inv. vec. field

$\because [X^\ell, Y^\ell](f_1 f_2) = X^\ell(f_1 \cdot Y^\ell(f_2) + Y^\ell(f_1)\cdot f_2)$

$- Y^\ell(f_1 X^\ell(f_2) + X^\ell(f_1) f_2)$

$= f_1 X^\ell Y^\ell (f_2) + X^\ell Y^\ell(f_1) f_2 - f_1 Y^\ell X^\ell(f_2) - Y^\ell X^\ell(f_1) f_2$

$= f_1 [X^\ell, Y^\ell](f_2) + [X^\ell, Y^\ell](f_1) f_2$ Leibniz rule

right invariance from that of X^ℓ, Y^ℓ.

$\rightsquigarrow [X^\ell, Y^\ell] = Z^\ell$ for some $Z \in T_e G$

$[X, Y] := Z$.

Rem if we use X^r, Y^r, we get $(-1)\times$ above cf. Prop.

- Lie groups with same Lie algs

$\Gamma \triangleleft G$ (normal) G connected $\Rightarrow \Gamma < Z(G)$

discrete $\Rightarrow G/\Gamma$ has the "same" tang. sp at e as G.

$T_e(G/\Gamma) = T_e G$. brackets are also same

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Universal cover (G: connected Lie grp)

concrete $\tilde{G}$ = { homotopy classes of paths from e }

$= \{ \gamma : [0,1] \to G,\ \gamma(0) = e \} /$ homotopy

abstract : $\tilde{G} \xrightarrow{p} G$ $\tilde{G}$ simply connected

p local homeo.

Group structure (for concrete)

"concatenate" paths

$\gamma' \cdot \gamma$: $\gamma(2t)$ $0 \le t \le \frac{1}{2}$

$\gamma'(2t-1)\gamma(1)$ $\frac{1}{2} \le t \le 1$

path from e to $\gamma'(1) \cdot \gamma(1)$

neutral elem: const path at e.

$\rightsquigarrow$ $\tilde{G}$ Lie grp, $\tilde{G} \to G$, $\gamma \mapsto \gamma(1)$ grp hom.

(local homeo)