Lie algs to Lie groups

· exponential map.

G : Lie group $\mathfrak{g} = T_eG$. tangent space at e

$X \in \mathfrak{g} \rightsquigarrow X^\ell$ right inv. vec. field $X^\ell_g(f) = X((R_g)^*(f))$

by general mfd. theory. $\exists\, \varepsilon > 0$, $\varphi_X(0) = e$

$\exists!$ $\varphi_X : (-\varepsilon, \varepsilon) \to G$ s.t. $\varphi_X'(t) = X^\ell(\varphi_X(t))$.

i.e. $\partial_t f(\varphi_X(t))|_{t=t_0} = X^\ell_{\varphi_X(t_0)}(f)$

Prop. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$ for $|s|, |t|, |s+t| < \varepsilon$

Proof. Fix t & consider deriv. at $s = s_0$

Left hand side $\partial_s f(\varphi_X(s)\varphi_X(t))|_{s=s_0}$

$= X^\ell_{\varphi(s_0)}(R_{\varphi_X(t)}{}^* f) = X^\ell_{\varphi_X(s_0)\varphi_X(t)}(f)$

Right hand side $\partial_s f(\varphi_X(s+t))|_{s=s_0} = X^\ell_{\varphi_X(s_0+t)}(f)$

$\Rightarrow$ as maps $(-\varepsilon', \varepsilon') \to G$ $\varepsilon' \ll 1$

$s \mapsto \varphi_X(s)\varphi_X(t)$ and $s \mapsto \varphi_X(s+t)$

are both $0 \mapsto \varphi_X(t)$, & has X^ℓ as "velocity"

$\underset{\text{uniqueness}}{\Rightarrow}$ they are same. ☒

Cor. φ_X extends to $\mathbb{R} \to G$ s.t. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$

$\because$ Fix $t \in \mathbb{R}$, take $N \gg 1$ s.t. $|\frac{t}{N}| < \varepsilon$.

for ε in Prop. $\rightsquigarrow$ Put $\varphi_X(t) = \varphi_X(\frac{t}{N})^N$.

Prop $\Rightarrow$ this is well def'ed & satisfies claim.

<u>exponential map</u> : $\exp : \mathfrak{g} \to G$ $X \mapsto \varphi_X(1)$

Ex. $G = SU(n)$ $\mathfrak{g} = su(n) = \{X \in M_n(\mathbb{C}) : X^* = -X\}$

$\varphi_X(t) = e^{tX} = I_n + tX + \frac{t^2}{2}X^2 + \cdots$

$\exp(X) = e^X = I_n + X + \cdots + \frac{1}{k!}X^k + \cdots$

$\because$ $e^{sX}e^{tX} = e^{(s+t)X}$ from comparison of coeffs.

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$(\varphi_X(t) = \exp(tX))_{t\in\mathbb{R}}$: one parameter group generated by X (in G)

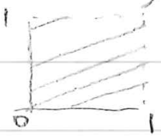
Lie subgroups.

N, M : manifolds, $j : N \to M$ inj. map (differentiable, ..) s.t. $j_\# : T_xN \to T_{f(x)}M$ is inj.

$X \mapsto (f \mapsto X(f\circ j))$

$\rightsquigarrow$ top of N does not have to be the same as the top. of img.

Ex. $\mathbb{R} \to \mathbb{T}^2$ (group hom!)



irrational slope

$\mathbb{T}^2$ has Lie alg $\mathbb{R}^2$, img of this emb. should corr. to $\{(t, \theta t) : t\in\mathbb{R}, \theta : \text{irr.}\}$ (subalg)

In the above situation, the img of j is called a immersed submfd.

Prop 2 G Lie group, $\mathfrak{g} = T_eG$ its Lie alg
$\mathfrak{h} \subset \mathfrak{g}$ Lie subalg
$\Rightarrow \exists$ immersed subgroup ($j : H \to G$ inj. hom) s.t. $\mathfrak{h} = T_eH$.

Proof : Step 1 H as a set $= \{h_1 \cdots h_k : h_i = \exp(X_i),\ X_i \in \mathfrak{h}\ \forall k\}$
this is a subgroup of G ($\exp(X)^{-1} = \exp(-X)$)

Step 2 $\exists$ neigh Δ of $0 \in \mathfrak{g}$ s.t. $G_0 = \exp(\Delta)$
$H_0 = \exp(\Delta \cap \mathfrak{h})$, satisfies $(H_0 \cdot H_0) \cap G_0 = H_0$
$H_0^{-1} = H_0$.

$\because$ We use Ado's theorem $\rightsquigarrow$ we may assume $G = GL_n(\mathbb{R})$

- $g \in G$ close to I_n ("$\| I_n - g \| < 1$")
  $\Rightarrow \log g = \log(I + (g - I)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(g-I)^n}{n}$
  is well-defied.
- $X, Y \in \mathfrak{h}$ small $\Rightarrow \exp(X)\exp(Y)$ close to I_n
- Baker-Campbell-Hausdorff formula

$$\log(\exp(X)\exp(Y))$$
$$= \sum_{n=0}^{\infty} \frac{1}{n} \sum_{\substack{r_1 + \cdots + s_1 + \cdots + s_{m-1} = n-1 \\ r_1 + s_1 \geq 1,\ r_2 + s_2 \geq 1 \cdots}} \frac{(-1)^{m-1}}{m} \left(\prod^{m-1} \frac{ad_X^{r_i}}{r_i!} \frac{ad_Y^{s_i}}{s_i!} \right) \frac{ad_X^{r_m}}{r_m!} (Y)$$
$$+ (X \leftrightarrow Y)$$

$$= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \cdots$$

$\|X\|, \|Y\|$ small $\Rightarrow$ this converges

cf. $-\log(2 - \exp(x+y)) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{r_i, s_i \\ r_i + s_i \geq 1}} \frac{x^{r_1}}{r_1!} \cdots\cdots \frac{y^{s_m}}{s_m!}$

Step 3. H becomes a Lie group by setting

$H_0 \underset{\log}{\xrightarrow{\sim}} \Delta \cap \mathfrak{h}$ open in $\mathfrak{h} \simeq \mathbb{R}^k$.

↑ make it a (open) neighborhood of e in H. ☒

Rem In the above, G connected
$\mathfrak{h}$ ideal in $\mathfrak{g} \Leftrightarrow H$ normal in G.

Th'm. G conn. Lie group $\mathfrak{g}$ its Lie alg
K another Lie grp, $\mathfrak{k}$ its Lie alg
$\varphi: \mathfrak{g} \to \mathfrak{k}$ Lie alg hom
Then $\exists$ Lie grp hom $\tilde{G} \xrightarrow{f} K$ inducing φ

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Conseq : $K = GL_n(\mathbb{R}) \rightsquigarrow \tilde{G} \to K$ rep. of $\tilde{G}$

$\mathfrak{g} \to \mathfrak{k} = \mathfrak{gl}_n(\mathbb{R})$ rep of $\mathfrak{g}$

so rep of $\tilde{G}$ $\equiv$ rep of $\mathfrak{g}$.

Proof of thm

Set $\mathfrak{h} = \{ (X, \varphi(X)) : X \in \mathfrak{g} \} \subset \mathfrak{g} \oplus \mathfrak{k}$.

(graph of φ).

Step 1 φ hom $\Leftrightarrow$ $\mathfrak{h}$ is a Lie subalg

Step 2 Regard $\mathfrak{g} \oplus \mathfrak{k}$ as the Lie alg of $\tilde{G} \times K$, H immersed subgroup corr. to $\mathfrak{h}$. Then $H \to \tilde{G}$ prj. to second factor is homeo.

$\because$ $\mathfrak{h} \to \mathfrak{g}$ is iso $\Rightarrow$ $H \to \tilde{G}$ is local homeo. $\underset{\tilde{G} \text{ simply conn.}}{\Rightarrow}$ $H \to \tilde{G}$ is homeo.

Step 3. Up to $H \cong \tilde{G}$ of Step 2,

The prj. to first factor induces φ. ∎