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Summary
- algebraic groups
- ideals in Lie algs.
  - derived series, lower central series
  - solvable / nilpotent algs

- Algebraic groups

K : (comm.) field

Algebraic group over K is given by
- alg. variety G over K

(separ. scheme of finite type over K, no nilpot in $\mathcal{O}_{G,g}$)
- morphisms $\operatorname{Spec} K \xrightarrow{e} G$, $G \times G \xrightarrow{M} G$, $G \xrightarrow{inv} G$.

satisfying the usual axioms of groups

Cartier) (Char $K = 0$) affine alg. grps $\equiv$ fin. gen. comm. Hopf alg over K

i.e. comm Hopf $\Rightarrow$ no nilpotents.

Ex.
- Elliptic curves : not affine
- (split) torus : $(K^\times)^n$
- $GL_n(K)$: $\mathcal{O}_{GL_n(K)} = K[(x_{ij})_{i,j=1}^n ; \det(x_{ij}) \neq 0]$

$$= K[(x_{ij})_{i,j}, \tilde{d}] / (\tilde{d} \times \det((x_{ij})_{i,j}) - 1)$$

- $SL_n(K)$: $\mathcal{O}_{SL_n(K)} = K[(x_{ij})_{i,j=1}^n] / (\det((x_{ij})_{i,j}) - 1)$

Why do we want to consider this?

Compact Lie group $G \rightsquigarrow$ alg. of "matrix coefficients" of $G \equiv \mathcal{O}_{G_{\mathbb{C}}}$ for the complexification $G_{\mathbb{C}}$ of G

Ex. $SU(n) \rightsquigarrow SL_n(\mathbb{C})$, $SO(n) \rightsquigarrow SO_n(\mathbb{C})$

- Ideals in Lie algs.

$\mathfrak{g}$: Lie alg.

<u>derived subalgebra</u> (commutator subalg)

$\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] =$ span of $[X, Y]$ $(X, Y \in \mathfrak{g})$

(also denoted by $\mathfrak{g}'$)

Prop 1 $\mathcal{D}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$

Proof) Almost by def of $\mathcal{D}(\mathfrak{g})$: $X, Y, Z \in \mathfrak{g}$

$\Rightarrow [X, [Y, Z]] \in \mathfrak{g}$ since $[Y, Z] \in \mathfrak{g}$ ∎

<u>Lower central series</u> $\mathcal{D}_n(\mathfrak{g})$: $\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$

$\mathcal{D}_{n+1}(\mathfrak{g}) = [\mathcal{D}_n(\mathfrak{g}), \mathfrak{g}]$

<u>Derived series</u> $\mathcal{D}^n(\mathfrak{g})$: $\mathcal{D}^1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$,

$\mathcal{D}^{n+1}(\mathfrak{g}) = [\mathcal{D}^n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g})] = \mathcal{D}(\mathcal{D}^n(\mathfrak{g}))$, $(\subset \mathcal{D}^{n+1}(\mathfrak{g}))$

Prop. 2 $\mathcal{D}_n(\mathfrak{g})$ and $\mathcal{D}^n(\mathfrak{g})$ are ideals of $\mathfrak{g}$

$(\Rightarrow \mathcal{D}_{n+1}(\mathfrak{g}) \subset \mathcal{D}_n(\mathfrak{g}),\ \mathcal{D}^{n+1}(\mathfrak{g}) \subset \mathcal{D}^n(\mathfrak{g}))$

Proof : By induction on n. (we'll check $\mathcal{D}_n(\mathfrak{g})$)

$n=1$) By Prop 1

general case) suppose $\mathcal{D}_n(\mathfrak{g}) \triangleleft \mathfrak{g}$,

$X, Z \in \mathfrak{g}$, $Y \in \mathcal{D}_n(\mathfrak{g})$

We want $[X, \underbrace{[Y, Z]}_{\mathcal{D}_{n+1}(\mathfrak{g})}] \in \mathcal{D}_{n+1}(\mathfrak{g})$

By Jacobi id. (& antisymm.)

$$[X,[Y,Z]] = \underbrace{[[Z,X],Y]}_{\text{in } [\mathfrak{g}, \mathcal{D}_n(\mathfrak{g})] = \mathcal{D}_{n+1}(\mathfrak{g})} + \underbrace{[\overbrace{[X,Y]}^{\mathcal{D}_n(\mathfrak{g}) \text{ by assumption}},Z]}_{\text{in } \mathcal{D}_{n+1}(\mathfrak{g})}.$$

∎

Rem $\mathcal{D}_n(\mathfrak{g}) / \mathcal{D}_{n+1}(\mathfrak{g})$ is in the center of $\mathfrak{g}/\mathcal{D}_{n+1}(\mathfrak{g})$

$\mathcal{D}^n(\mathfrak{g}) / \mathcal{D}^{n+1}(\mathfrak{g})$ is commutative

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$\mathfrak{g}$ is <u>solvable</u> if $\mathcal{D}^k(\mathfrak{g}) = 0$ for $k \gg 1$

<u>nilpotent</u> if $\mathcal{D}_k(\mathfrak{g}) = 0$ for $k \gg 1$

$\mathcal{D}^k(\mathfrak{g}) \subset \mathcal{D}_k(\mathfrak{g}) \rightsquigarrow$ nilpot. $\Rightarrow$ solv.

Ex. "$ax+b$"-alg. $\left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} : a, b \in K \right\} = \mathfrak{g}$

$$\left[\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \right] = \begin{bmatrix} 0 & 2(ab'-ba') \\ 0 & 0 \end{bmatrix}$$

Char $K \neq 2 \Rightarrow \mathcal{D}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in K \right\}$

$\mathcal{D}^2(\mathfrak{g}) = 0$, $\mathcal{D}_2(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$ $\quad (= \mathcal{D}_k(\mathfrak{g}) \quad k \geq 1)$

so $\mathfrak{g}$ is solvable but not nilpotent

Ex. $\mathfrak{g}$ comm $\Leftrightarrow \mathcal{D}(\mathfrak{g}) = 0$ $\quad (\Rightarrow$ nilpot.$)$

Solvable algs have "flexible" (and complicated) reps.

Ex. $\mathfrak{g} = K^2$ (comm.)

Rep of $\mathfrak{g}$ on $V \equiv$ commuting transforms

$T_1 = \pi_{(1,0)}, \quad T_2 = \pi_{(0,1)} \in \mathrm{End}(V)$

T_1 can be normalized using Jordan normal form

$\rightsquigarrow T_2$ on each gen. eigenspace of T_1, with compatibility with "flag" structure...

Prop 3 $\mathfrak{g}$ is solvable iff $\exists\ \mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0$

s.t. $\mathfrak{g}_{n+1} \triangleleft \mathfrak{g}_n$, $\mathfrak{g}_n / \mathfrak{g}_{n+1}$ comm.

Proof. $\Rightarrow$: we can take $\mathfrak{g}_n = \mathcal{D}^n(\mathfrak{g})$

$\Leftarrow$: $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ comm $\Leftrightarrow \mathfrak{g}_{n+1} \supset \mathcal{D}(\mathfrak{g}_n)$

$\rightsquigarrow$ By induction $\underset{\searrow 0}{\mathfrak{g}_n} \supset \mathcal{D}^n(\mathfrak{g})$ ∎

Cor. $\mathfrak{g} \triangleright \mathfrak{h}$: The followings are equiv:

1) $\mathfrak{g}$ is solvable 2) $\mathfrak{h}$ & $\mathfrak{g}/\mathfrak{h}$ are solv.

Proof $\Rightarrow$: $(\mathfrak{g}_n)_n$ as in Prop 3

$\rightsquigarrow$ $\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{g}_n$, $\bar{\mathfrak{g}}_n$ = img of $\mathfrak{g}_n$ in $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$

do the job.

$\Leftarrow$: $(\mathfrak{h}_n)_n$ for $\mathfrak{h}$, $(\bar{\mathfrak{g}}_m)_m$ for $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$ as in Prop 3, $\bar{\mathfrak{g}}_N = 0$

$\rightsquigarrow$ $\mathfrak{g}_m$: inv. img of $\bar{\mathfrak{g}}_m$ in $\mathfrak{g}$. $(m \leq N)$ so $\mathfrak{g}_N = \mathfrak{h}$.

$\mathfrak{g}_{N+k} = \mathfrak{h}_k$ do the job ∎

Thm There is a (unique) largest solvable ideal of $\mathfrak{g}$.

Proof. $\mathfrak{a}, \mathfrak{b}$: solv. ideal of $\mathfrak{g}$ $\Rightarrow$ so is $\mathfrak{a}+\mathfrak{b}$

Enough to prove ($\Rightarrow$ span of all solv. ideals will be the max. one)

$\mathfrak{a} \lhd \mathfrak{a}+\mathfrak{b}$, $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} = \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$

$\rightsquigarrow$ Use Cor. to $\mathfrak{a} \to \mathfrak{a}+\mathfrak{b} \to \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$. ∎

$\mathrm{Rad}(\mathfrak{g})$, <u>solvable</u> <u>radical</u> of $\mathfrak{g}$: the largest solvable ideal of $\mathfrak{g}$.

$\mathfrak{g}$ is <u>semisimple</u> if there is no (nonzero) solvable ideal of $\mathfrak{g}$ ($\mathrm{Rad}(\mathfrak{g}) = 0$)

Rem. Char $K = 0$ $\Rightarrow$ $\forall$ semisimple $\mathfrak{g}$ is $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, $\mathfrak{g}_i$ simple.