Summary
- Structure of solvable Lie algs
  - Engel's th'm & Lie's th'm.
- Killing form
  - Cartan's criterion

- Structure of solvable Lie algs

Motto : solvable Lie algs are like upper triangular matrices

Engel's theorem) V: vec. sp over K,
$\mathfrak{g} \subset \mathfrak{gl}(V)$ Lie subalg s.t. $\forall X \in \mathfrak{g}$ is a nilpotent endomor. of V ($X^n = 0$ $n \gg 1$)
Then $\exists\, 0 \neq v \in V$ s.t. $\forall X \in \mathfrak{g}$ $Xv = 0$

Cor. $\exists$ basis $(v_1, v_2, \cdots, v_n)$ of V s.t. $\forall X \in \mathfrak{g}$ is "strictly upper triangular"
i.e. $Xv_1 = 0$, $Xv_2 \in Kv_1$, $\cdots$, $Xv_k \in \langle v_1, \cdots, v_{k-1} \rangle$, $\cdots$

Ex. $\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} : a,b,c \in K \right\} \subset \mathfrak{gl}_3(K)$ $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Proof of Cor. Induction on $\dim V =: n$.
S1. Set $v_1 = v$ from Th'm, $V' = V/Kv_1$
$\rightsquigarrow$ $\mathfrak{g}$ acts on V' $X[w + v_1] = [Xw + Xv_1] = [Xw]$

S2. Ind. hypo. $\Rightarrow$ V' has basis $[v_2], \cdots, [v_n]$ s.t. $\forall X \in \mathfrak{g}$ is repr'ed by str. up. triang. mat.
$X[v_k] \in \langle [v_2], \cdots, [v_{k-1}] \rangle$ means $Xv_k \in \langle v_1, \cdots, v_{k-1} \rangle$
$v_1, \cdots, v_n$ basis of V.

Proof of Th'm
Step 1. X nilpot end. on V $\Rightarrow$ ad_X nilpot. end. on $\mathfrak{gl}(V)$
$\because$ $X^k = 0$ $\Rightarrow$ $\underbrace{\mathrm{ad}_X^{2k+1}(T)}_{\text{linear comb. of monoms } X^a T X^{2k-a}} = 0$

Step 2. Prove claim by ind. on $m = \dim \mathfrak{g}$.
Step 2-1 $\exists\, \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m - 1$
$\because$ Take any max. proper subalg as $\mathfrak{h}$

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$\mathfrak{h}$ is invar. under ad_X for $X \in \mathfrak{h}$.

$\Rightarrow \overline{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}/\mathfrak{h}$. , ad_X nilp (Step 1) $\Rightarrow \overline{ad}_X$ also.

Ind. hypo $\Rightarrow \exists 0 \neq \overline{Y} \in \mathfrak{g}/\mathfrak{h}$ s.t. $\overline{ad}_X(\overline{Y}) = 0$ $\forall X \in \mathfrak{h}$.

i.e. $\exists Y \in \mathfrak{g} \setminus \mathfrak{h}$ $\forall X \in \mathfrak{h}$ $[X, Y] \in \mathfrak{h}$

So $\mathfrak{h}' = \mathfrak{h} + K \cdot Y$ is a subalg, $\mathfrak{h} \triangleleft \mathfrak{h}'$

maximality of $\mathfrak{h}$ $\Rightarrow \mathfrak{h}' = \mathfrak{g}$.

Step 2-2 $\mathfrak{h}$ as in Step 2-1. , take $Y \in \mathfrak{g} \setminus \mathfrak{h}$.

Ind. hypo $\Rightarrow$ $W = \{ v \in V : \forall X \in \mathfrak{h} \ X v = 0 \} \neq 0$

Enough to show $\exists v \in W : Y v = 0$

$\Leftarrow YW \subset W$ $\because$ Y is a nilpotent end.

Step 2-3 $YW \subset W$

$\because$ $XYv = YXv + [X, Y] v$.

$X \in \mathfrak{h}$, $v \in W$ $\Rightarrow YXv = 0$, $[X, Y] \in \mathfrak{h} \Rightarrow [X, Y] v = 0$

so $XYv = 0$ for $\forall X \in \mathfrak{h}$. $\boxtimes$

Lie's theorem. K alg. closed (like $\mathbb{C}$). V : K-vec. sp.

$\mathfrak{g} \subset \mathfrak{gl}(V)$ solvable subalg.

$\Rightarrow \exists 0 \neq v \in V$ eigenvector for any $X \in \mathfrak{g}$.

Cor. $\exists$ basis $(v_1, \cdots, v_n)$ of V, s.t. $\forall X \in \mathfrak{g}$ is "upper triangular" $\begin{bmatrix} \lambda_1 & * & * \\ & \lambda_2 & * \\ 0 & & \ddots \end{bmatrix}$

Proof of Th'm. Ind. on $m = \dim \mathfrak{g}$.

Step 1 $\exists \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m - 1$

$\because$ $\mathcal{D}^k \mathfrak{g} \searrow 0$ by assumption. so $\mathcal{D}(\mathfrak{g}) \neq \mathfrak{g}$

$\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ comm. $\Rightarrow$ any subsp. is ideal.

$\rightsquigarrow$ we can take inv. img. of codim 1 subsp. in $\mathfrak{g}/\mathcal{D}(\mathfrak{g})$.

Step 2 Ind. hypo $\Rightarrow \exists 0 \neq v \in V$ eigenvec. for $\forall X \in \mathfrak{h}$.

Set $Xv = \lambda(X) v$ $X \in \mathfrak{h}$ $\lambda : \mathfrak{h} \to K$ lin.

Step 3 $W = \{ v' \in V : \forall X \in \mathfrak{h} \ X v' = \lambda(X) v' \}$

$Y \in \mathfrak{g} \setminus \mathfrak{h}$.

5.3-1 Enough to show $YW \subset W$

$\because$ K alg. closed $\Rightarrow$ Y has eigenvec. in W.

Step 3-2 $YW \subset W \Leftrightarrow \lambda([X,Y]) = 0 \quad \forall X \in \mathfrak{h}$.

$\because$ Again $XYv' = YXv' + [X,Y]v'$

$X \in \mathfrak{h}$, $v' \in W$ $\Rightarrow$ LHS is $\lambda(X)Yv' + \lambda([X,Y])v'$

so claim $\Leftrightarrow \lambda([X,Y]) = 0$

Step 4 $0 \neq w \in W \Rightarrow U_w = \langle w, Yw, Y^2 w, \cdots \rangle$

$X \in \mathfrak{h} \Rightarrow XY^k w \in \lambda(X) Y^k w + \langle w, \cdots, Y^{k-1} w \rangle$

$\because$ induction on k, $XY^k w = YXY^{k-1}w + \underbrace{[X,Y]}_{\text{in } \mathfrak{h}} Y^{k-1} w$

Step 5 $\lambda([X,Y]) = 0$

$\lambda([X,Y])$: "diagonal" entries of $[X,Y]|_{U_w}$.

$= \frac{1}{\dim U_w} \mathrm{Tr}([X,Y]|_{U_w})$.

But X, Y pres $U_w \Rightarrow \mathrm{Tr}([X,Y]|_{U_w}) = \mathrm{Tr}([X|_{U_w}, Y|_{U_w}]) = 0$

$\boxtimes$

∘ Killing form.

Recall K alg clos. V : K-vec. sp. $X \in \mathrm{End}(V)$

$\Rightarrow X = X_s + X_n$ $\quad X_s$: diagonalizable

X_n : nilpotent.

$X_s X_n = X_n X_s$. this decomp is unique.

(from Jordan normal form).

V : K-vec. sp. <u>Killing form</u> (assoc. to V) is

$B_V(X,Y) = \mathrm{Tr}(XY) \quad X, Y \in \mathrm{End}(V)$.

$\mathfrak{g}$: Lie alg. the <u>Killing form</u> on $\mathfrak{g}$ is

$B(X,Y) = B_{\mathfrak{g}}(\mathrm{ad}_X, \mathrm{ad}_Y)$

(⚠ careful when $\mathfrak{g} = \mathfrak{gl}(V)$)

$K \subset \mathbb{C}$ 2018.09.25 no 4

Cartan's criterion) $\mathfrak{g} < \mathfrak{gl}(V)$ $B_V(X, Y) = 0$
for $X, Y \in \mathfrak{g}$ $\Rightarrow$ $\mathfrak{g}$ solvable

Idea : $\mathcal{D}(\mathfrak{g})$ nilpot $\Rightarrow$ $\mathfrak{g}$ solv.

Step 1 Engel's thm $\Rightarrow$ enough to check $\forall X \in \mathcal{D}(\mathfrak{g})$
is nilpot end. on V.
$\Leftrightarrow$ eigenvals (λ_i) of X are all 0

Step 2 $D = X_s$ claim $\Leftrightarrow$ $\underbrace{\mathrm{Tr}(\bar{D} X)}_{= \sum |\lambda_i|^2} = 0$

$X = \sum [Y_i, Z_i]$ $\Rightarrow$ $\bar{D} X = \sum \bar{D} [Y_i, Z_i]$
$\mathrm{Tr}(\bar{D} X) = \sum \mathrm{Tr}([\bar{D}, Y_i] Z_i)$
$\Rightarrow$ Enough to have $[\bar{D}, Y_i] \in \mathfrak{g}$.

Step 3. $\mathrm{ad}\,\bar{D}$ poly in $\mathrm{ad}\, X$.
$\because$ $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ &
$\mathrm{ad}(\bar{D})$ poly of $\mathrm{ad}(D)$

Cor. $B(\mathfrak{g}, \mathfrak{g}) = 0$ $\Rightarrow$ $\mathfrak{g}$ solvable
$\because$ $\underset{\text{comm.}}{\mathfrak{z}(\mathfrak{g})} \to \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) = \text{img of } \mathfrak{g} \text{ in } \mathrm{End}(\mathfrak{g})$
solvable from Cartan's criterion.

Cor $\mathfrak{g}$ solvable $\Leftrightarrow$ $B(\mathfrak{g}, \mathcal{D}(\mathfrak{g})) = 0$ $(K = \mathbb{C})$
$\Rightarrow$: Lie's th'm $\Rightarrow$ $\exists$ basis of $\mathfrak{g}$ s.t. ad_X
is uppertriangular. $\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{bmatrix}$

$\rightsquigarrow \left[\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}, \begin{bmatrix} \mu_1 & & * \\ & \mu_2 & \\ 0 & & \ddots \end{bmatrix} \right] = \begin{bmatrix} 0 & & * \\ & 0 & \\ 0 & & \ddots \end{bmatrix}$

$\Rightarrow$ $\mathrm{Tr}(\mathrm{ad}_X\, \mathrm{ad}_{[Y,Z]}) = 0$.

$\Leftarrow$: $B(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{g})) = 0$ $\Rightarrow$ $\mathcal{D}(\mathfrak{g})$ solvable
$\mathcal{D}(\mathfrak{g}) \to \mathfrak{g} \to \underset{\text{comm.}}{\mathfrak{g}/\mathcal{D}(\mathfrak{g})}$