Summary

- Semisimple Lie algebras
  - complete reducibility ; unitary trick
  - Nondegeneracy of Killing form.

- Semisimple Lie algs.

! Generally reps of Lie algs don't have complete reducibility. (and don't behave well with "$X = D + N$")

Ex $\mathfrak{g} = K \overset{\pi}{\curvearrowright} V = K^2$ by $\pi_t = \begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix}$ $(t \in K)$

$V' = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in K \right\}$ is the only nontriv. invar. subsp. $\Rightarrow$ doesn't have complement

also $\begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}}_{\text{diag.}} + \underbrace{\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}}_{\text{nilpot.}}$

is not of the form π_s

$\rightsquigarrow$ But all is good for semisimple algs

(Recall : semisimple $\equiv$ no solvable ideals)

$\mathfrak{g}$: Lie alg over K, $K \subset \mathbb{C}$.

Prop. $\mathfrak{g}$ is semisimple $\Leftrightarrow$ Killing form B is nondegenerate $(\forall X \neq 0 \, \exists Y \, B(X,Y) \neq 0)$

Step 1 $\mathfrak{s} = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ B(X,Y) = 0 \}$ is ideal of $\mathfrak{g}$

$\because$ $B([Z,X],Y) + B(X,[Z,Y]) = 0$ (invariance)

implies $X \in \mathfrak{s}, Z \in \mathfrak{g} \Rightarrow [Z,X] \in \mathfrak{s}$

Invariance : LHS $= \mathrm{Tr}(\mathrm{ad}_{[Z,X]} \mathrm{ad}_Y + \mathrm{ad}_X \mathrm{ad}_{[Z,Y]})$

$= \mathrm{Tr}\left((\mathrm{ad}_Z \mathrm{ad}_X - \cancel{\mathrm{ad}_X \mathrm{ad}_Z})\mathrm{ad}_Y + \mathrm{ad}_X(\cancel{\mathrm{ad}_Z \mathrm{ad}_Y} - \mathrm{ad}_Y \mathrm{ad}_Z)\right)$

$= \mathrm{Tr}(\mathrm{ad}_Z \mathrm{ad}_X \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y \mathrm{ad}_Z) = 0$

(cf. "Step 2" of Cartan's criterion)

Step 2 "$\Rightarrow$" of the claim.

$\mathrm{ad}_{\mathfrak{s}}$: img of $\mathfrak{s}$ in $\mathrm{End}(\mathfrak{g})$. $B(\mathrm{ad}_{\mathfrak{s}}, \mathrm{ad}_{\mathfrak{s}}) = 0$

Cartan's criterion $\Rightarrow$ $\mathrm{ad}_{\mathfrak{s}}$ is solvable

$\underset{\text{comm.}}{\mathfrak{Z}(\mathfrak{g}) \cap \mathfrak{s}} \to \mathfrak{s} \to \underset{\text{solv.}}{\mathrm{ad}_{\mathfrak{s}}}$ $\Rightarrow$ $\mathfrak{s}$ also solvable (ideal)

$\mathfrak{g}$ s.s. $\Rightarrow$ $\mathfrak{s} = 0$ i.e. B is nondegen.

Step 3 $\mathfrak{h} \triangleleft \mathfrak{g} \Rightarrow \mathcal{D}(\mathfrak{h}) \triangleleft \mathfrak{g}$

$\because$ $[X,[Y,Z]] = [Z,[X,Y]] + [Y,[Z,X]]$

from (antisymm &) Jacobi identity

$Y, Z \in \mathfrak{h}$, $X \in \mathfrak{g}$ $\Rightarrow$ right hand side is in $\mathcal{D}(\mathfrak{h})$

Step 4 $\mathrm{Rad}(\mathfrak{g}) \neq 0 \Rightarrow \mathfrak{g}$ contains a comm. ideal.

$\because$ $\mathrm{Rad}(\mathfrak{g})$ solvable $\Rightarrow$ $\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})) \searrow 0$.

if $\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})) \neq 0$ and $\mathcal{D}^{k+1}(\mathrm{Rad}(\mathfrak{g})) = 0$

$\mathfrak{a} = \mathcal{D}^k(\mathrm{Rad}(\mathfrak{g}))$ is comm. $\mathcal{D}''(\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})))$

Step 5 "$\Leftarrow$" of the claim. (works for any K)

Want : B nondeg $\Rightarrow$ $\mathfrak{g}$ has no nonzero comm. ideal

suppose $\mathfrak{a} \triangleleft \mathfrak{g}$ comm. $X \in \mathfrak{a}$.

$Y \in \mathfrak{g}$ $\Rightarrow$ $\mathrm{ad}_X \mathrm{ad}_Y$ $\quad \mathfrak{g} \xrightarrow{\mathrm{ad}_X} \underset{\text{ideal}}{\mathfrak{a}}$, $\underset{\text{comm.}}{\mathfrak{a}} \xrightarrow{\mathrm{ad}_X} 0$

so matrix pres looks like
$$\begin{array}{cc} \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} & \begin{array}{l}) \ \mathfrak{a} \\ ) \ \text{complement} \end{array} \\ \begin{array}{cc} \mathfrak{a} & \text{compl.} \end{array} & \end{array}$$

$\Rightarrow \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) = 0$

So $B(X,Y) = 0$ if $X \in \mathfrak{a}$ $\underset{\text{nondeg.}}{\Longrightarrow}$ $X = 0$. ∎

Ex. $\mathfrak{g} = \mathfrak{sl}_2(K)$ $\quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ad_E : $E \mapsto 0$, $F \mapsto H$, $H \mapsto -2E$

ad_F : $E \mapsto -H$, $F \mapsto 0$, $H \mapsto 2F$

ad_H : $E \mapsto 2E$, $F \mapsto -2F$, $H \mapsto 0$

2018.09.26 no.3

so $B(X,Y)$ is

$Y \backslash X$	E	F	H
E	0	4	0
F	4	0	0
H	0	0	8

Cor $\mathfrak{g}$ semisimple $\Leftrightarrow$ $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $\mathfrak{g}_i$ simple

$\because$ $\Rightarrow$: $\mathfrak{h} \triangleleft \mathfrak{g}$, set $\mathfrak{h}^\perp = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{h} \ B(X,Y)=0\}$

$\mathfrak{h} \cap \mathfrak{h}^\perp$ solv. by Cartan's criterion

$\mathfrak{g}$ s.s. $\Rightarrow$ $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$.

invar. of B $\Rightarrow$ $\mathfrak{h}^\perp$ also ideal.

$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as Lie alg.

Keep decomposing. $\boxtimes$

- Complete reducibility of semisimple Lie algs

(π, V): rep of $\mathfrak{g}$, $W \subset V$: $\mathfrak{g}$-invar. subsp.

$\mathfrak{g}$ semisimple $\Rightarrow$ $\exists\, W' \subset V$ $\mathfrak{g}$-inv. complement

Analytic proof for $K = \mathbb{C}$ (Weyl's unitary trick)

Sketch for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

1 $(\mathfrak{su}_n)_\mathbb{C} = \mathfrak{sl}_n\mathbb{C}$ ($\mathfrak{sl}_n\mathbb{C}$ is the complexification of $\mathfrak{su}_n$)

so $\mathfrak{sl}_n\mathbb{C} \overset{\pi}{\curvearrowright} V \equiv \mathfrak{su}_n \overset{\pi}{\curvearrowright} V$ ( )
by cplx lin. transforms

2 $\mathfrak{su}_n \overset{\pi}{\curvearrowright} V$ by cplx lin. $\equiv$ $SU(n) \overset{\tilde\pi}{\curvearrowright} V$ by cplx lin.
i.e. rep. of $SU(n)$ on V

3 $SU(n)$ is compact $\Rightarrow$ $\exists$ $\tilde\pi$-invar. Herm. inn. prod. on V.

$\because$ Haar measure μ on $SU(n)$ $\int f(gh)\, d\mu(g) = \int f\, d\mu$

$(v,v')_0$ any inn. prod $\rightsquigarrow$ $(v,v') = \int (\tilde\pi_g v, \tilde\pi_g v')_0\, d\mu(g)$
is invar.

2018. 09. 26 no. 4

4. $W \subset V$ invar. $\Rightarrow W^\perp = \{ v \in V : \forall v' \in W \ (v, v') = 0 \}$
is $\tilde{\pi}$ - invar.

5. $W^\perp$ is a rep of su_n. (by cplx lin. transf.)

6. $W^\perp$ is also a rep of $sl_n(\mathbb{C})$.

Algebraic proof (still requires $K \subset \mathbb{C}$) $\mathfrak{g} \curvearrowright V$

Step 0 we may assume $\mathfrak{g} \subset \mathfrak{gl}(V)$
(img of semisimple is still semisimple)

Step 1. B_V is nondeg. on $\mathfrak{g}$

$\because$ $\mathfrak{s} = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ B_V(X, Y) = 0 \}$ ideal

$\rightsquigarrow$ $\mathfrak{s}$ solvable by Cartan's criterion

$\Rightarrow$ should be 0 by semisimplicity.

Step 2 $(X_i)_{i=1}^n$ basis of $\mathfrak{g}$, $(X^i)_{i=1}^n$ dual basis of $\mathfrak{g}$ rel. to B_V $B_V(X_i, X^j) = \delta_{ij}$

$C_V = \sum X_i X^i \in \mathrm{End}(V)$ <u>Casimir</u> operator

$\Rightarrow C_V$ is an intertwiner.

$\because$ $[Y, C_V] = \sum_i [Y, X_i] X^i + X_i [Y, X^i]$

$= \sum_{i,j} B_V([Y, X_i], X^j) X_j X^i + B_V(X_j, [Y, X^i]) X_i X^j$

invariance of B_V $\Rightarrow$ RHS is 0.
& relabel i & j

Step 3 Claim for $\dim W = \dim V - 1$, W irred.

- $C_V|_W$ is scalar (eigenspace would be subrep)
- $\mathfrak{g} \curvearrowright V/W$ is trivial ($\mathfrak{g} = \mathcal{D}(\mathfrak{g})$ acts trivially)
- $\mathrm{Tr}(C_V) = \sum B_V(X_i, X^i) = \dim(\mathfrak{g})$

$\Rightarrow C_V|_W = \alpha \, \mathrm{Id}_W$, $\alpha \neq 0$, $V = W \oplus \mathrm{Ker}(C_V)$

Step 4 Claim for $\dim W = \dim V - 1$

Ind. on $\dim W$.

$0 \neq Z \subset W$ inv. $\xrightarrow[\text{ind.}]{} V/Z \simeq W/Z \oplus Y_{1\text{-}\dim}$.

$\tilde{Y}$: inv. img of Y splits as $Z \oplus Y_1$.

$\rightsquigarrow$ $V \simeq W \oplus Y_1$

Step 5 dim W general, W irred

$\mathrm{End}_{\mathfrak{g}}(W)$ is 1-dim., triv. rep. of $\mathfrak{g}$.

$$\begin{array}{ccc} \mathrm{Hom}_K(V, W) & \xrightarrow{\text{res.}} & \mathrm{End}_K(W) \\ \cup & & \cup \\ U & \longrightarrow & \mathrm{End}_{\mathfrak{g}}(W) \end{array}$$

surjective., $\mathfrak{g}$ acts by ad

surj.

U: inv. img of $\mathrm{End}_{\mathfrak{g}}(W)$

$\mathrm{Ker}(\mathrm{res}|_U)$ is codim 1. $\mathfrak{g}$-inv. subsp

$\Rightarrow$ $U \simeq \mathrm{Ker}(\mathrm{res}|_U) \oplus U_0$. U_0 : triv.

i.e. $\mathrm{Hom}_K(V, W)$ has $\mathfrak{g}$-inv. elem which restr. to Id_W.

this is $V \overset{\varphi}{\to} W$ prj. intertwiner.

$V = W \oplus \mathrm{Ker}(\varphi)$. as $\mathfrak{g}$-rep.