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Summary
- semisimple - nilpotent decomposition in semisimple Lie algs
- representation of $\mathfrak{sl}_2(\mathbb{C})$

∘ Semisimple - nilpot decomp.

K : alg. closed field (or perfect field ; Char $K = 0$, $\mathbb{F}_p, \dots$)

V : fin. dim vec. sp. over K.

$T \in \mathrm{End}(V) \rightsquigarrow T = \underset{\text{diag'ble}}{T_s} + \underset{\text{nilpotent}}{T_n}$

T_s, T_n polynoms in T. ($\Rightarrow$ commute with each other)

$\mathfrak{g}$: Lie alg over K. $\mathfrak{g} \overset{\pi}{\curvearrowright} V$

! generally $\not\exists\, Z, Y \in \mathfrak{g}$ $\pi(X)_s = \pi(Y)$, $\pi(X)_n = \pi(Z)$.

Th'm $\mathfrak{g}$ semisimple, (π, V) rep of $\mathfrak{g}$, $X \in \mathfrak{g}$

$\Rightarrow \exists Y, Z \in \mathfrak{g}$ $\pi(X)_s = \pi(Y)$, $\pi(X)_n = \pi(Z)$

If π is faithful, Y & Z are indep. of π.

Key Lem. $\mathfrak{g} \subset \mathfrak{gl}(V)$ semisimple $X \in \mathfrak{g}$

$\Rightarrow X_s, X_n \in \mathfrak{g}$

Proof of Lem

Step 1. For $W \subset V$ put $S_W = \{Y \in \mathfrak{gl}(V) : YW \subset W,\ \mathrm{Tr}(Y|_W) = 0\}$

and $\tilde{\mathfrak{g}} = \{Y \in \mathfrak{gl}(V) : [Y, \mathfrak{g}] \subset \mathfrak{g}\}$ (normalizer)

Then $\mathfrak{g} = \tilde{\mathfrak{g}} \cap \left(\bigcap_{W : \mathfrak{g}\text{-inv}} S_W \right)$

$\because$ $\subset$ is obvious from def. Put $\mathfrak{g}'$ = right hand side

$\mathfrak{g} \triangleleft \tilde{\mathfrak{g}} \Rightarrow \mathfrak{g} \triangleleft \mathfrak{g}'$. so $\mathfrak{g}$ acts on $\mathfrak{g}'$ by ad

Compl. reducibility $\Rightarrow \mathfrak{g}' = \mathfrak{g} \oplus U$ for ad - inv. U. We want $U = 0$.

Take $Y \in U$ Enough to show : $W \subset V$ irred subrep. of $\mathfrak{g}$ $\Rightarrow Y|_W = 0$

($\because V = W_1 \oplus \cdots \oplus W_k$ irr. decomp)

$Y \in U \Rightarrow [\mathfrak{g}, Y] \in \mathfrak{g} \cap U = 0 \Rightarrow Y$ is intertwiner

Schur's lemm. $\Rightarrow Y|_W$ is scalar.

$\mathrm{Tr}(Y|_W) = 0 \Rightarrow Y|_W = 0$.

$\leadsto$ goal is to show $X_s \in \tilde{\mathfrak{g}}$, $X_s \in S_W$, etc.

Step 2 $W \subset V$ $\mathfrak{g}$-inv $\Rightarrow X_s, X_n \in S_W$

$\because$ $\mathfrak{g}$ semisimple $\Rightarrow \mathfrak{g} = \mathcal{D}(\mathfrak{g})$ (use $\mathfrak{g} = \oplus \mathfrak{g}_i$)

$\Rightarrow \mathrm{Tr}(X|_W) = \mathrm{Tr}(\Sigma [Y_i|_W, Z_i|_W]) = 0$

$X_n|_W$ is nilpot $\Rightarrow \mathrm{Tr}(X_n|_W) = 0$

$X_s|_W = (X - X_n)|_W$ also 0-trace.

Step 3 $X_s, X_n \in \tilde{\mathfrak{g}}$

$\because$ $\left.\begin{matrix} \mathrm{ad}_{X_s} \\ \mathrm{ad}_{X_n} \end{matrix}\right)$ on $\tilde{\mathfrak{g}}$ is $\begin{pmatrix} \text{semisimple} \\ \text{nilpot.} \end{pmatrix}$ part of ad_X

$\because$ $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{gl}(V))$ stabilizes $\mathfrak{g}$. of X

ad_{X_s} is diagonalizable ($v_1, v_2, \dots$ eigenvec.)

$E_{ij} v_k = \delta_{jk} v_i \Rightarrow E_{ij}$ eigenvec. of ad_{X_s}

ad_{X_n} is nilpot. (see 09.25)

$\mathrm{ad}_{X_s} \mathrm{ad}_{X_n} = \mathrm{ad}_{X_n} \mathrm{ad}_{X_s}$, $\mathrm{ad}_X = \mathrm{ad}_{X_s} + \mathrm{ad}_{X_n}$

$\boxed{\mathrm{ad}_{X_s} \text{ stab } \mathfrak{g}}$

so ad_{X_s} is a polynom. of ad_X

$\mathrm{ad}_X(\mathfrak{g}) \subset \mathfrak{g} \Rightarrow \mathrm{ad}_{X_s}(\mathfrak{g}) \subset \mathfrak{g}$. $\boxed{2}$

Proof of thm. $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ $\mathfrak{g}' = \mathrm{Im}\,\pi$.

Step 1. $\mathfrak{g} \simeq \ker \pi \oplus \mathfrak{g}'$ as Lie alg. ($\Rightarrow \mathfrak{g}'$ semisimpl)

$\because$ $\ker \pi$ ideal. $\mathfrak{g}' \simeq (\ker \pi)^\perp$ for Killing form (see 09.26)

Step 2. $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{g})$ is faithful

$\Rightarrow \exists Y, Z$ $\mathrm{ad}_Y = (\mathrm{ad}_X)_s$, $\mathrm{ad}_Z = (\mathrm{ad}_Z)_n$

$\because$ Lem.

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Step 3 $\pi(Y) = \pi(X)_s$, $\pi(Z) = \pi(X)_n$

$\because$ $\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'}$ semisimple, $[\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'}, \mathrm{ad}\,\pi(Z)|_{\mathfrak{g}'}] = 0$

$\mathrm{ad}\,\pi(Z)|_{\mathfrak{g}'}$ nilpot.

$\mathrm{ad}\,\pi(X)_s$, $\mathrm{ad}\,\pi(X)_n$ have same prop.

Uniqueness of ss-nilpot. dec.

$\Rightarrow$ $\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'} = \mathrm{ad}\,\pi(X)_s|_{\mathfrak{g}'}$, etc.

so $\pi(Y) - \pi(X)_s \in \mathfrak{z}(\mathfrak{g}') = 0$. ∎

Ex. $\mathfrak{sl}_2(\mathbb{C})$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ from deriv. of $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} = e^{tH}$

$t \in \sqrt{-1}\,\mathbb{R} \Rightarrow e^{tH}$ unitary

(π, V) rep of V $\Rightarrow$ $\pi(e^{tH})$ diag'ble

$\Rightarrow$ $\pi(H)$ also diag'ble.

- Representation of $\mathfrak{sl}_2(\mathbb{C})$. (or $\mathfrak{sl}_2(K)$ Char $K = 0$)

$\mathfrak{sl}_2(\mathbb{C}) = \langle H, E, F \rangle$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (simple Lie alg)

$\rightsquigarrow$ in the <u>defining</u> <u>rep</u> on $\mathbb{C}^2$

$H = H_s$, $E = E_n$, $F = F_n$.

$\Rightarrow$ $\forall$ rep (π, V) $\pi(H)$: diag'ble.

$\pi(E)$, $\pi(F)$: nilpotent.

Weight dec.

Prop. $\lambda \in \mathbb{C}$ $\pi(H)v = \lambda v$ $\Rightarrow$ $v' = \pi(E)v$, $v'' = \pi(F)v$

satisfy $\pi(H)v' = (\lambda+2)v'$, $\pi(H)v'' = (\lambda-2)v''$

Proof $[\pi(H), \pi(E)] = \pi([H, E]) = 2\pi(E)$

$\Rightarrow$ $\pi(H)\pi(E)v - \pi(E)\underbrace{\lambda v}_{\pi(H)v} = 2\pi(E)v$ ∎

$v \in V$ is a <u>highest weight vector</u> (of wght λ) if

- it is an eigenvector of $\pi(H)$; $\pi(H)v = \lambda v$
- $\pi(E)v = 0$

Prop. $v \in V$ h.w.vec. $\Rightarrow W = \langle v, \pi(F)v, \pi(F)^2 v, \cdots \rangle$

is an $\mathfrak{sl}_2(\mathbb{C})$-inv. subsp.

Proof $\pi(F)$-inv : obvious

$\pi(H)$-invar : prev. Prop $\Rightarrow$ $\pi(F)^k v$ eigenvec

$\pi(E)$-invar : $[\pi(E), \pi(F)] = \pi([E,F]) = \pi(H)$

By induction $\pi(E)\pi(F)^{k+1} v \in \langle \pi(F)^k v \rangle$.

$k=0$: $\pi(E)\pi(F)v - \pi(F)\underbrace{\pi(E)v}_{0} = \lambda v$

gen. $\pi(E)\pi(F)^{k+1} v - \pi(F)\pi(E)\pi(F)^k v$

$= \pi(H)\pi(F)^k v$.

Weight decomposition.

$V = \bigoplus_\alpha V_\alpha$. $V_\alpha = \langle v \in V : \pi(H)v = \alpha v \rangle$.

Prop. v h.w.v. of wght λ $\Rightarrow \pi(F)^k v = 0$ for $k \gg 1$

only if $\lambda \in \mathbb{N}$

Proof. Claim $\pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$.

$\because$ induction on k, $[\pi(E), \pi(F)] = \pi(H)$.