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Summary

- Representation of $sl_2(\mathbb{C})$, cont'd
  - irred. reps
  - rep. of $SU(2)$
  - fusion rules.

- Rep'n of $sl_2\mathbb{C}$ cont'd.

$sl_2(\mathbb{C}) = \langle H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \rangle$

(π, V) : fin. dim rep of $sl_2(\mathbb{C})$

$\rightsquigarrow$ • $V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$, $V_\alpha = \{ v \in V : \pi(H) v = \alpha v \}$

weight sp. decomp. ($V_\alpha = 0$ for almost all α)

• $\pi(E) V_\alpha \subset V_{\alpha+2}$, $\pi(F) V_\alpha \subset V_{\alpha-2}$

highest weight vec v (of wght λ) : $0 \neq v \in V_\lambda \cap \operatorname{Ker} \pi(E)$.

$\rightsquigarrow$ $W = \langle \pi(F)^k v : k = 0, 1, 2, \cdots \rangle$ $sl_2(\mathbb{C})$-inv.

Prop. Under the above setting $\lambda \in \mathbb{N} = \{0, 1, 2, \cdots\}$

$\lambda = \max n$
$\pi(F)^n v \neq 0$

Proof. Step 1 $\pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$ for $k \geq 1$

$\because$ Induction on k (also makes sense for $k=0$)

$k = 1$: $\pi(E)\pi(F) v - \pi(F) \underbrace{\pi(E) v}_{0} = \pi(\underbrace{[E, F]}_{H}) v = \lambda v$

general : $\pi(E)\pi(F)^{k+1} v - \pi(F) \underbrace{\pi(E)\pi(F)^k v}_{k(\lambda-k+1)\pi(F)^{k-1} v \text{ by ind. hyp.}} = \pi(H)\pi(F)^k v$

$\Rightarrow \pi(E)\pi(F)^{k+1} v = (k(\lambda - k + 1) + (\lambda - 2k))\pi(F)^k v$

$= (k+1)(\lambda - k)\pi(F)^k v$.

Step 2 take $k \geq 1$ s.t. $\pi(F)^k v = 0 \neq \pi(F)^{k-1} v$.

then $0 = \pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$

$\Rightarrow \lambda = k - 1$. ∎

Suppose (π, V) is irreducible. ($\Rightarrow W = V$).

$V = V_\lambda \oplus V_{\lambda-2} \oplus \cdots \oplus V_{-\lambda}$

$V_{\lambda - 2k} = \langle \pi(F)^k \cdot v \rangle$. 1-dim.

"Step 1" above determines how $\pi(E)$ should act. (and $\pi(H)$, $\pi(F)$ are already determined)

$\Rightarrow$ Irreducible rep. of $\mathfrak{sl}_2(\mathbb{C})$ are classified by the highest weight $\lambda = 0, 1, 2, \cdots$ (corresponding rep: $(\lambda+1)$-dimensional).

Concrete realization

$V^{(n)} = \langle x^n, x^{n-1}y, \cdots, y^n \rangle$ space of homogeneos polynoms in x & y. $(n+1)$-dim.

$\pi^{(n)}(E)f = y\partial_x f$, $\pi(F)f = x\partial_y f$ $\quad f \in V^{(n)}$

$\pi^{(n)}(H)f = y\partial_y f - x\partial_x f$

$\rightsquigarrow [\pi^{(n)}(E), \pi^{(n)}(F)] = \pi^{(n)}(H)$, $[\pi^{(n)}(H), \pi^{(n)}(E)] = 2\pi^{(n)}(E)$ etc.

$\pi^{(n)}(H)(x^{n-k}y^k) = 2k - n$.

$\Rightarrow y^n$: highest wght vec. of wght n.

Rep of $SU(2)$.

(cplx) rep. of $\mathfrak{sl}_2(\mathbb{C}) \leftrightarrow$ cplx rep. of $\mathfrak{su}_2$

$\leftrightarrow$ (f.d.) unitary rep. of $SU(2)$

$\rightsquigarrow V^{(n)}$ are the irred. unitary reps of $SU(2)$

generic elem of $SU(2)$: $\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}$ $|a|^2+|c|^2=1$

$$\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - \bar{c}y \\ cx + \bar{a}y \end{bmatrix}$$

invariant inn. prod. $(x^{n-k}y^k, x^{n-j}y^j) = \delta_{jk}\binom{n}{k}^{-1}$

$\sqrt{-1}\,H = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} \in \mathfrak{su}_2$ $\quad (= \{X \in M_2(\mathbb{C}) : X^* = -X\})$

$\exp(2\pi t\sqrt{-1}H) = \begin{bmatrix} e^{2\pi t\sqrt{-1}} & 0 \\ 0 & e^{-2\pi t\sqrt{-1}} \end{bmatrix} \in SU(2)$

this is periodic. (same val $t + \mathbb{Z}$)

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$\rightsquigarrow$ any rep. π of $SU(2)$ should induce $\pi(H)$ with integer eigenvals.

$\because$ $\pi(e^{2\pi\sqrt{-1}H}) = e^{2\pi\sqrt{-1}\pi(H)}$ should be $\pi(I_2) = Id_V$.

Fusion rules of $sl_2(\mathbb{C})$.

(π, V), (π', V') rep of $sl_2(\mathbb{C})$

$\rightsquigarrow$ we want to understand $(\pi\otimes\pi', V\otimes V')$

(irred. decomp.)

We may assume π, π' are irred

($(\pi_1 \oplus \pi_2) \otimes \pi' \simeq (\pi_1 \otimes \pi') \oplus (\pi_2 \otimes \pi')$, etc.)

Task:

1 do weight decomp. of $V\otimes V'$

$$V\otimes V' \simeq \bigoplus_{m,n} \underbrace{V_m \otimes V'_n}_{\text{weight } (m+n)} = \bigoplus_k \bigoplus_m V_m \otimes V'_{k-n}$$

$$(\pi\otimes\pi')(X) = \pi(X)\otimes Id_{V'} + Id_V \otimes \pi'(X)$$

$$(\pi\otimes\pi')(H)(v\otimes v') = mv\otimes v' + v\otimes nv' = (m+n)v\otimes v'$$

Ex 1) $(\pi, V) = (\pi', V')$: def. rep on $\mathbb{C}^2$ $(= V^{(1)})$

$$V \otimes V = (V_1 \otimes V_1) \oplus (V_{-1}\otimes V_1 \oplus V_1 \otimes V_{-1}) \oplus (V_{-1}\otimes V_{-1})$$

weight 2 0 -2

$v_1 \in V_1$ h.w. vec. $\Rightarrow$ $v_1\otimes v_1$ h.w.vec. in $V\otimes V$.

2 Find highest weigt vec. ξ in $V\otimes V'$

3 Take $W = \langle \xi, \pi(F)\xi, \pi(F)^2\xi, \dots \rangle$

4 Take $W^\perp$, repeat from 2 (find $\xi' \in W^\perp, \dots$)

$v_1\otimes v_1$, $(\pi\otimes\pi)(F)(v_1\otimes v_1) = v_1\otimes v_{-1} + v_{-1}\otimes v_1$

$(\pi\otimes\pi)(F)^2(v_1\otimes v_1) = v_{-1}\otimes v_{-1}$

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span irred. subrep $W \simeq (\pi^{(2)}, V^{(2)})$

$\langle v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle$ is the complement (triv. rep)

$\Rightarrow \pi^{(1)} \otimes \pi^{(1)} \simeq \pi^{(2)} \oplus \pi^{(0)}$

$\quad\quad\quad\quad\quad Sym^2(V) \quad \Lambda^2(V)$

2) $\pi^{(1)} \otimes \pi^{(k)} \simeq \pi^{(k+1)} \oplus \pi^{(k-1)}$

again v_1, $v^{(k)}_k$ h.w. vecs in $V^{(1)}$, $V^{(k)}$

$\Rightarrow v_1 \otimes v^{(k)}_k$ h.w. vec of whght $k+1$

in $V^{(1)} \otimes V^{(k)} \rightsquigarrow$ get a irred. subrep. W

isom. to $V^{(k+1)}$

$(V^{(1)} \otimes V^{(k)})_{k-1}$ is 2-dim

$(V^{(1)}_1 \otimes V^{(k)}_{k-2}) \oplus (V^{(1)}_{-1} \otimes V^{(k)}_k)$

$\rightsquigarrow W^{\perp}$ has wght $(k-1)$-comp.

$\rightsquigarrow$ get a copy of $V^{(k-1)}$

$\dim V^{(k+1)} \oplus V^{(k-1)} = k+2+k$

$\dim V^{(1)} \otimes V^{(k)} = 2(k+1)$

$\Rightarrow$ We are done.

3) $\pi^{(m)} \otimes \pi^{(n)} \simeq \pi^{(m+n)} \oplus \pi^{(m+n-2)} \oplus \cdots \oplus \pi^{(|m-n|)}$

Rem. $U_n(x)$ Chebyshev polynom. of second kind

$U_0(x) = 1$, $U_1(x) = 2x$, $U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x)$

$\sum U_n(x) t^n = \dfrac{1}{1 - 2tx + t^2}$

$U_n(\frac{x}{2})$ basis of $\mathbb{Z}[x]$.

Rep. ring of $sl_2(\mathbb{C})$ $R(sl_2(\mathbb{C})) = \langle [\pi^{(n)}] : n \in \mathbb{N} \rangle$

with $\oplus$, $\otimes$ as ring structure

$\simeq \mathbb{Z}[x]$ by $[\pi^{(n)}] \leftrightarrow U_n(\frac{x}{2})$