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Summary

- Representation of $\mathfrak{sl}_3(\mathbb{C})$.
  - Structure of $\mathfrak{sl}_3(\mathbb{C})$: Cartan subalg, root decomp.
  - eigenvalues, weight lattice, innerproduct

Structure of $\mathfrak{sl}_3(\mathbb{C}) = \{X \in M_3(\mathbb{C}) : \operatorname{Tr} X = 0\}$

Want : convenient "representatives" of diagonalizable / nilpotent elems.

Diagonalizable ones $\rightsquigarrow$ take diag matrices

Nilpotent ones $\rightsquigarrow$ take strictly upper / lower triangular mats

$\mathfrak{h} = \{X \in \mathfrak{sl}_3(\mathbb{C}) : \text{diag. matrix}\} = \langle H_1, H_2 \rangle$

$H_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$.

Rem $\mathfrak{h}$ is maximal diagonalizable commutative "Cartan subalgebra"

$\because$ Y comm. with $\forall X \in \mathfrak{h}$ $\Rightarrow$ Y comm. with any diag. $\Rightarrow$ Y diag.

extra : scalar mats

$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, ...

Rem. $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathbb{C} E_{ij} \right)$ "eigendecomp. for $\mathfrak{h}$"

- E_{ij} is a joint eigenvector for elems of $\mathfrak{h}$.

e.g. $[H_1, E_{12}] = \left[\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] = 2 E_{12}$

$[H_2, E_{12}] = \left[\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] = -E_{12}$

- $\mathfrak{h}$ max comm. $\Leftrightarrow$ $\mathfrak{h}$ is the joint ker of $\operatorname{ad}_{\mathfrak{h}}$

- $\mathfrak{sl}_2$ - triples

Lem. $i \neq j$ $\Rightarrow$ E_{ij}, E_{ji}, $H_{ij} = [E_{ij}, E_{ji}]$ span a subalg isom to $\mathfrak{sl}_2$

Pf. $H_{ij} = E_{ii} - E_{jj}$. $[H_{ij}, E_{ij}] = 2E_{ij}$, $[H_{ij}, E_{ji}] = -2E_{ji}$

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from $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$

Notn. $s_{ij} = \langle E_{ij}, E_{ji}, H_{ij} \rangle$. $(= s_{ji})$.

Rem $H_1 = H_{12}$, $H_2 = H_{23}$

(π, V) : rep. of $\mathfrak{sl}_3(\mathbb{C})$ $\Rightarrow$ $\pi|_{s_{ij}} : \mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\sim} s_{ij} \curvearrowright V$ rep. of $\mathfrak{sl}_2(\mathbb{C})$.

- Irred. decomp of $\pi|_{s_{ij}}$ has $(\pi^{(n)}, V^{(n)})$ ($(n+1)$-dim irrep) as direct summands
- $\pi^{(n)}(H)$ has eigenvals $n, n-2, \cdots, -n \in \mathbb{Z}$

$\Rightarrow$ $\pi(H_{ij})$ have integer eigenvalues.

$v \in V$ joint eigenvec. for $\mathfrak{h}$. $\rightsquigarrow$ $\pi(X) v = \lambda(X) v$.

- $\lambda(X)$ is linear in X $\lambda \in \mathfrak{h}^* \ (\simeq \mathbb{C}^2)$
- $\lambda(H_1), \lambda(H_2) \in \mathbb{Z}$. i.e. λ belongs to a subgrp $\simeq \mathbb{Z}^2$ (lattice in $\mathfrak{h}^*$)

$\Lambda_W = \{\lambda \in \mathfrak{h}^* : \lambda(H_1), \lambda(H_2) \in \mathbb{Z}\}$ weight lattice.

Example defining rep $V = \mathbb{C}^3$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ joint eigenvecs for $\mathfrak{h}$

$H_1 e_1 = e_1$, $H_2 e_1 = 0$, $H_1 e_2 = -e_2$, $H_2 e_2 = e_2$, $\cdots$

$\lambda_1(H_1) = 1$, $\lambda_1(H_2) = 0$ $\lambda_2(H_1) = -1$, $\lambda_2(H_2) = 1$, $\lambda_3(H_1) = 0$, $\lambda_3(H_2) = -1$

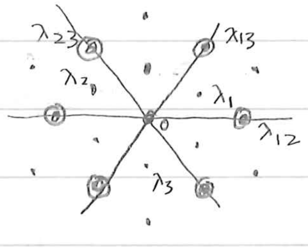
(so $\lambda_3 = -(\lambda_1 + \lambda_2)$, (λ_1, λ_2) basis of Λ_W, $\cdots$

Eigenvalues of $\mathrm{ad}\ \mathfrak{sl}_3(\mathbb{C}) \curvearrowright \mathfrak{sl}_3(\mathbb{C})$ span a sublattice of Λ_W root lattice Λ_R.

$[X, E_{ij}] = \lambda(X) E_{ij}$ $X \in \mathfrak{h}$

$\rightsquigarrow$ $\lambda_{12}(H_1) = 2$, $\lambda_{12}(H_2) = -1$, $\lambda_{23}(H_1) = -1$, $\lambda_{23}(H_2) = 2$, $\cdots$

$(\lambda_{12} = \lambda_1 - \lambda_2)$ $(\lambda_{23} = \lambda_1 + 2\lambda_2)$



$\cdot$: weight lattice

$\odot$: roots

$\Lambda_W / \Lambda_R \simeq \mathbb{Z}/3\mathbb{Z}$, $\cdots$

- inner product

Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$

$\rightsquigarrow$ restrict to $\mathfrak{h}$.

Prop. B is (symm.) pos. def. on $\langle H_1, H_2 \rangle_{\mathbb{R}}$.

Proof 1) $B(X,Y) = 6\,\mathrm{Tr}(XY)$ for $\mathfrak{sl}_3(\mathbb{C})$

H_1, H_2 are real symm $\Rightarrow$ pos. def.

Proof 2) $B(X,Y) = \sum_{\alpha:\text{root}} \alpha(X)\alpha(Y)$ $\quad X, Y \in \mathfrak{h}$

$= \sum_{i \neq j} \lambda_{ij}(X)\lambda_{ij}(Y)$

$\because$ ad_X is $\lambda_{ij}(X)$ on $\mathbb{C}E_{ij}$

0 on $\mathfrak{h}$

$\Rightarrow$ $\mathrm{ad}_X \mathrm{ad}_Y$ is $\lambda_{ij}(X)\lambda_{ij}(Y)$ on $\mathbb{C}E_{ij}$

0 on $\mathfrak{h}$.

$\alpha(H_i) \in \mathbb{Z}$ $\Rightarrow$ $\alpha(X)^2 \geq 0$. $\Rightarrow$ $B(X,X) \geq 0$

$\therefore B(X,X) = 0$ means $\alpha(X) = 0$ for $\alpha = \lambda_{ij}$

$\Rightarrow \mathrm{ad}_X = 0 \Rightarrow X \in \mathfrak{z}(\mathfrak{sl}_3(\mathbb{C})) = 0$ $\quad \boxtimes$

$\langle H_1, H_2 \rangle_{\mathbb{R}}^* = \mathbb{R}\Lambda_W$ also has symm. pos. def inner prod.

(E : Euclidean sp. $\rightsquigarrow$ $E \overset{\phi}{\cong} E^*$ by $\phi_v(w) = (v, w)$

$\rightsquigarrow$ E^* has inn. prod by $(\phi_v, \phi_w) = (v, w)$)

$(X,Y) = \mathrm{Tr}(XY) = \frac{1}{6} B(X,Y)$ has matrix

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ for basis (H_1, H_2)

$\rightsquigarrow$