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Summary
- rep. of $\mathfrak{sl}_3(\mathbb{C})$, cont'd.
  - root vector action & weight decomp
  - positive roots, highest wght vecs
  - Weyl group

- Root vectors, wght decomp.

$E_{ij}$ : matrix units $\rightsquigarrow$ $\lambda_{ij}$ corr. roots.

(e.g. $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda_{12}\Big(\underbrace{\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}}_{H_1 = H_{12}}\Big) = 2$, $\lambda_{12}\Big(\underbrace{\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}}_{H_2 = H_{23}}\Big) = -1$)

$(\pi, V)$ : rep $\rightsquigarrow$ weight decomp. $V = \bigoplus_{\lambda \in \Lambda_W} V_\lambda$

$$V_\lambda = \{ v \in V : \pi(X)v = \lambda(X)v \quad (X \in \mathfrak{h}) \}$$

Lem. $\pi(E_{ij}) V_\lambda \subset V_{\lambda + \lambda_{ij}}$

Proof Take $v \in V_\lambda$.

Want : $\pi(X)\pi(E_{ij})v = (\lambda(X) + \lambda_{ij}(X))\pi(E_{ij})v$ for $X \in \mathfrak{h}$.

$$\left( \begin{array}{l} \pi(X)\pi(E_{ij}) - \pi(E_{ij})\pi(X) = \pi([X, E_{ij}]) = \lambda_{ij}(X)\pi(E_{ij}) \\ \pi(X)v = \lambda(X)v. \end{array} \right.$$

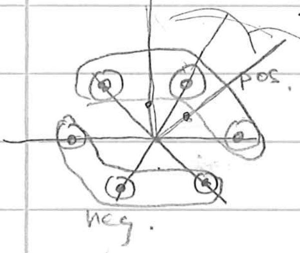
$$\Rightarrow \pi(X)\pi(E_{ij})v = \lambda_{ij}(X)\pi(E_{ij})v + \pi(E_{ij})\underbrace{\pi(X)v}_{\lambda(X)v} \qquad \square$$

Def. $\lambda_{12}, \lambda_{23}, \lambda_{13}$ : <u>positive roots</u>

$\lambda_{21}, \lambda_{32}, \lambda_{31}$ : <u>negative</u> roots.

$(\pi, V)$ rep, $v \in V_\lambda \cap \operatorname{Ker}\pi(E_{12}) \cap \operatorname{Ker}\pi(E_{23})$ :

a <u>highest weigh vecto</u>



highest weight will be here.

Rem. $v \in \operatorname{Ker}\pi(E_{12}) \cap \operatorname{Ker}\pi(E_{23})$

$\Rightarrow v \in \operatorname{Ker}\pi(E_{13}) \quad \because E_{13} = [E_{12}, E_{23}]$

Prop. We can always find a h.w.vec.

Proof. $L(\lambda) = (\lambda, \lambda_{13})$ "length in 60° direction"

$V = \bigoplus_\lambda V_\lambda$ pick $\lambda_0$ with

- $V_{\lambda_0} \neq 0$
- $L(\lambda_0)$ maximal among such.

$\Rightarrow L(\lambda + \lambda_{12}),\ L(\lambda + \lambda_{23}) > L(\lambda)$

$\because (\lambda_{12}, \lambda_{13}),\ (\lambda_{23}, \lambda_{13})$ are pos.

so $\pi(E_{12}) V_{\lambda_0} \subset V_{\lambda_0 + \lambda_{12}} = 0$, etc.

Thm. $(\pi, V)$ rep., $v \in V_\lambda$ h.w.vec.

$W = \langle \pi(E_{21})^{a_1} \pi(E_{32})^{b_1} \cdots \pi(E_{21})^{a_k} \pi(E_{32})^{b_k} v$

$a_i, b_i \geq 0 \rangle$ is a subrep.

Proof Step 1 $W$ is closed under $\pi(E_{ji})$ $i<j$

$\because$ obvious for $E_{21}, E_{32}$.

$\Rightarrow E_{31} = [E_{32}, E_{21}]$ also preserves $W$.

Step 2 $W$ is closed under $\pi(H_1), \pi(H_2)$

$\because \pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v \in V_{\lambda + (\sum a_i)\lambda_{21} + (\sum b_i)\lambda_{32}}$

by Lem $\Rightarrow \pi(H_i)$ is acting by scalar mult.

Step 3. $W$ is closed under $\pi(E_{ij})$ $i<j$

$\because$ Induction on "word length" $\sum a_i + \sum b_i$

"length 0": $\pi(E_{12}) v = 0 = \pi(E_{23}) v$

general: if $a_1 > 0$: $[E_{12}, E_{21}] = H_1 \Rightarrow$

$\pi(E_{12}) \pi(E_{21}) = \pi(E_{21}) \pi(E_{12}) + \pi(H_1)$

so $\pi(E_{12}) \pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v$

$= \pi(E_{21}) \underbrace{\pi(E_{12}) \underbrace{\pi(E_{21})^{a_1 - 1} \cdots \pi(E_{32})^{b_k} v}_{\text{len. 1 less}}}_{\text{in } W}$

$+ \underbrace{\pi(H_1) \pi(E_{21})^{a_1 - 1} \cdots \pi(E_{32})^{b_k} v}_{\text{in } W} \in W.$

$[E_{23}, E_{21}] = 0 \Rightarrow \pi(E_{32}) \pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v$

$= \pi(E_{21}) \underbrace{\pi(E_{32}) \pi(E_{21})^{a_1 - 1} \cdots \pi(E_{32})^{b_k} v}_{\text{in } W}$

If $a_1 = 0$ : switch the role of $E_{12}$ & $E_{23}$ $\square$

Cor $(\pi, V)$ irred $\Rightarrow$ $V$ is generated by
- h.w. vec. $v \in V_\lambda$ • successive application of $\pi(E_{21})$, $\pi(E_{32})$.

( in fact : highest weight $\lambda$ completely determines $V$ cf. $\mathfrak{sl}_2$ case )

- Weyl group.

Recall rep of $\mathfrak{sl}_3(\mathbb{C}) \Longleftrightarrow$ rep of $SL_3(\mathbb{C})$

$\rightsquigarrow$ weight decomp of $V$

$\leftrightarrow$ decomp. as rep. of $H = \{g \in SL_3(\mathbb{C}) : \text{diag}\}$

( $H = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & b^{-1} \end{bmatrix} ; a, b \in \mathbb{C}^\times \right\}$ cplx 2-torus )

<u>Normalizer</u> $N = N_{SL_3(\mathbb{C})}(H) = \{ g \in SL_3(\mathbb{C}) : gHg^{-1} = H \}$.

Ex. $g_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \in N$

1) $g_1 \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & b^{-1} \end{bmatrix} g_1^{-1} = \begin{bmatrix} a^{-1}b & & \\ & a & \\ & & b^{-1} \end{bmatrix}$, $g_2 \begin{bmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1} \end{bmatrix} g_2^{-1} = \begin{bmatrix} a & & \\ & b^{-1} & \\ & & a^{-1}b \end{bmatrix}$

2) $g_1^2 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$, $g_2^2 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \in H$.

$g_1 g_2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow (g_1 g_2)^3 = I_3$.

$\Rightarrow N/H \simeq S_3$

<u>Weyl group</u> of $\mathfrak{sl}_3(\mathbb{C})$ : $W = N/H$. ; generators $s_i = [g_i]$

$[g]h = ghg^{-1}$ $(g \in N, h \in H)$ defs an action of $W$ on $H$ ( $[gh']h = gh'hh'^{-1}g^{-1} = [g]h$ )

$\Rightarrow$ induces an action on $\mathfrak{h}$ (= Lie alg of $H$).

$\begin{cases} s_1 H_1 = -H_1 \\ s_1 H_2 = H_1 + H_2 \end{cases} \leftrightarrow \begin{cases} s_1 \begin{bmatrix} a & & \\ & a^{-1} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} a^{-1} & & \\ & a & \\ & & 1 \end{bmatrix} \\ s_1 \begin{bmatrix} 1 & & \\ & b & \\ & & b^{-1} \end{bmatrix} = \begin{bmatrix} b & & \\ & 1 & \\ & & b^{-1} \end{bmatrix} \end{cases}$

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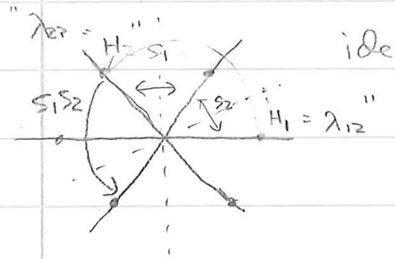
$s_2 H_1 = H_1 + H_2$, $s_2 H_2 = -H_2$




identifying $\langle H_1, H_2 \rangle_{\mathbb{R}}$ with $\mathbb{R}\Lambda_W$ by inn. prd

so $s_i$ acts by reflection along orthog. compl. of $H_i$

W acts on $\mathfrak{h}^*$, $\mathbb{R}\Lambda_W$, $\Lambda_W$, $\Lambda_R$ in natural way $[g]\lambda = \lambda([g]^{-1} \sim)$.

Prop. $(\pi, V)$ rep of $\mathfrak{sl}_3(\mathbb{C})$. $\mathrm{supp}(V) = \{\lambda \in \Lambda_W : V_\lambda \neq 0\}$

$\Rightarrow$ $\mathrm{supp}(V)$ is stable under $W$.

Proof. $\pi_g$ induces a rep of $SL_3(\mathbb{C})$.

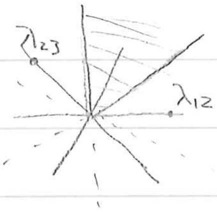
Claim: $\pi(g) V_\lambda = V_{[g]\lambda}$.

$\because$ $\pi(X)\pi(g)v = \pi(g) \underbrace{\pi(g^{-1}) \pi(X) \pi(g)}_{\pi(g^{-1}Xg)} v$

$= \pi(g_i)\, \pi([g]^{-1}(X))\, v = \lambda([g]^{-1}(X))\, \pi(g_i) v$

$= ([g]\lambda)(X)\, \pi(g_i) v$. ∎

Fundamental dom of $W = S_3 \curvearrowright \mathbb{R}\Lambda_W$




highest weights belong here