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summary
- Root system
  - axioms
  - canonical copies of $\mathfrak{sl}_2$.

- Root system.

Goal (after some time...)

complex simple Lie alg. $\mathfrak{g}$, $\mathfrak{h}$: Cartan subalg

$\rightsquigarrow$ roots (eigenvals of $\mathrm{ad}\,\mathfrak{h} \curvearrowright \mathfrak{g}$) ; Killing form on $\mathfrak{g} \rightsquigarrow$ on $\mathfrak{h}, \mathfrak{h}^*$ $\rightsquigarrow$ root system

$\rightsquigarrow$ Dynkin diagram encoding $\begin{pmatrix}\text{length}\\ \text{angle}\end{pmatrix}$ of roots

$\rightsquigarrow$ classification of cplx simple Lie algs

Recap. $\mathfrak{g}$: simple Lie alg. over $\mathbb{C}$ (e.g. $\mathfrak{sl}_n(\mathbb{C})$)

$\mathfrak{h}$: Cartan subalg. of $\mathfrak{g}$ maximal diagonalizable commutative ($X, Y \in \mathfrak{h} \Rightarrow [X, Y] = 0$

$\exists$ faithful rep / $\forall$ rep π $\forall X \in \mathfrak{h}$ $\pi(X)$ diag'ble)

e.g. $\mathfrak{h} = \{X \in \mathfrak{sl}_n(\mathbb{C}) : \text{diag}\} < \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$ root decomp

$\alpha \in \mathfrak{h}^*$ "root" $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{h}\ [Y, X] = \alpha(Y)X\}$

Fact 1. (for $\alpha \neq 0$) $\dim \mathfrak{g}_\alpha \leq 1$.

2. $\mathfrak{g}_\alpha \neq 0 \Rightarrow \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] =: S_\alpha$ is a subalg of $\mathfrak{g}$, isom to $\mathfrak{sl}_2(\mathbb{C})$

choose $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$, $H_\alpha := [E_\alpha, F_\alpha]$

(π, V) rep of $\mathfrak{g} \Rightarrow V = \bigoplus_\beta V_\beta$

$\beta \in \mathfrak{h}^*$ "weight" $V_\beta = \{v \in V : \forall Y \in \mathfrak{h}\ \pi_Y v = \beta(Y)v\}$

$\{0\} \cup$ roots $=$ weights of $\pi = \mathrm{ad}$

$\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$ & knowing weights of $\mathfrak{sl}_2$

$\Rightarrow$ $\forall$ wght β, $\beta(H_\alpha) \in \mathbb{Z}$.

wght lattice $\Lambda_W = \{ \beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z} \}$

root lattice Λ_R subgrp of Λ_W generated by roots.

$E = \mathbb{R} \cdot \Lambda_W$. (real) dual sp. of $\mathfrak{h}_0 = \langle H_\alpha : \alpha \text{ root} \rangle_{\mathbb{R}}$

Killing form $B_{\mathfrak{g}}(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$

- nondeg. on $\mathfrak{g}$ ($\Leftarrow$ Cartan's criterion)
- invariance $B_{\mathfrak{g}}([Z, X], Y) + B_{\mathfrak{g}}(X, [Z, Y]) = 0$

$\Rightarrow$ ($B_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0$ iff $\alpha = -\beta$ (incl. $\alpha = 0$)
nondeg. on $\mathfrak{h}$ $(= \mathfrak{g}_0)$)

Fact 4 $X \in \mathfrak{h}_0 \Rightarrow B_{\mathfrak{g}}(X, X) \geq 0$, $= 0$ only if $X = 0$

Write (v, w) $(v, w \in E)$ for the Euclidean inn. prod. on E.

Fact 3 α : root $\Rightarrow$ $T_\alpha = \dfrac{2}{B_\alpha(H_\alpha, H_\alpha)} H_\alpha$ is the elem

s.t. $B_{\mathfrak{g}}(T_\alpha, H) = \alpha(H)$

(img of iso $\mathfrak{h}^* \to \mathfrak{h}$ corr. to $B_{\mathfrak{g}}$)

α, β roots $\Rightarrow$ $\dfrac{2B(\alpha, \beta)}{B(\alpha, \alpha)} = \alpha(H_\beta) \in \mathbb{Z}$.

Fact 5 α root. $k \in \mathbb{Z} \setminus \{\pm 1\} \Rightarrow k\alpha$ not root.

$s_\alpha(v) = v - \dfrac{2(\alpha, v)}{(\alpha, \alpha)} \alpha$ reflection along $\alpha^\perp$.

Rem. (π, V) rep. β weight $\Rightarrow \bigoplus_n V_{\beta + n\alpha}$ subrep of $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

$\Rightarrow$ $\mathrm{supp}(V) = \{ \beta : \text{wght } V_\beta \neq 0 \}$ is closed under s_α

$\because$ $\mathfrak{sl}_2(\mathbb{C})$ rep. has "symmetric" wghts.

$\Rightarrow$ $R = \{\beta : \text{root}\} \subset E$ is closed under s_α.

$W = \langle s_\alpha : \alpha : \text{root} \rangle \subset O(E)$ <u>Weyl group</u>

Fact : R span E ($\mathbb{R}\Lambda_R = \mathbb{R}\Lambda_W$)

<u>Axiom</u> of root system.

E : Euclidean sp. ($\mathbb{R}$-vec. sp. & pos. def. inn. prd)

$R \subset E$ is a <u>root</u> <u>system</u> if

1) $|R| < \infty$, $\mathbb{R}$-span of R is E

2) $\alpha \in R$, $k \in \mathbb{Z}$ $\Rightarrow$ $k\alpha \in R$ iff $k = \pm 1$

3) $s_\alpha(v) = v - \dfrac{2(\alpha, v)}{(\alpha, \alpha)}\alpha$ maps R to R

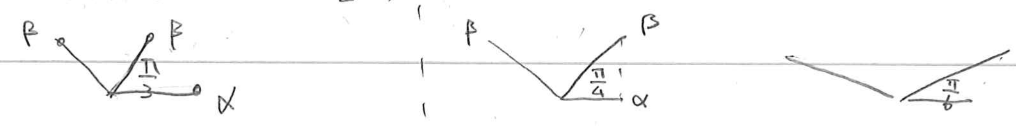
4) $n_{\alpha,\beta} = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Rem $n_{\alpha,\beta}\, n_{\beta,\alpha} = 4\cos^2\theta$ for θ : angle between α & β.

$4\cos^2\theta \in \mathbb{Z}$ $\Rightarrow$ $\cos\theta = \pm\dfrac{\sqrt{a}}{2}$ $a = 0, 1, 2, 3$.

$a = 1$ ($\cos\theta = \pm\frac{1}{2}$) $a = 2$ $a = 3$

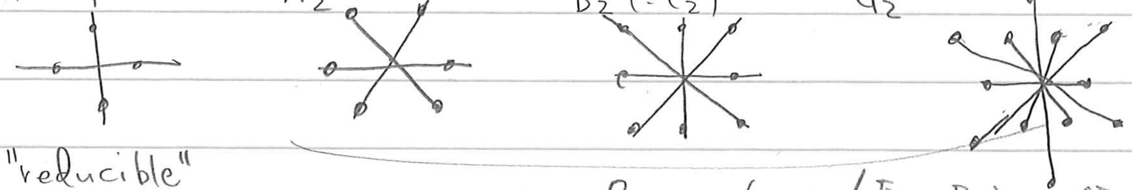
($\|\beta\| \geq \|\alpha\|$)



$\|\alpha\| = \|\beta\|$ $\|\beta\| = \sqrt{2}\,\|\alpha\|$ $\|\beta\| = \sqrt{3}\,\|\alpha\|$.

Ex. $\dim E = 2$

$A_1 \times A_1$ A_2 $B_2 (= C_2)$ G_2



"reducible"

irred. (not $(E_1, R_1) \times (E_2, R_2)$).

Motto : simple Lie alg $\rightsquigarrow$ irred root system, "remembers everything"

Proof of Fact 6 $\quad \mathbb{R}\Lambda_R = \mathbb{R}\Lambda_W$

Equivalent to. $\mathbb{C}\Lambda_R = \mathfrak{h}^*$.

Take $X \in (\mathbb{C}\Lambda_R)^\perp \subset \mathfrak{h}$ (we want $X = 0$)

$[X, Y] = \alpha(X) Y = 0$ for all $Y \in \mathfrak{g}_\alpha$

$\Rightarrow [X, Y] = 0$ for all $Y \in \mathfrak{g}$. $\Rightarrow X \in \mathfrak{z}(\mathfrak{g}) = 0$

Proof of Facts 1 & 2

Step 1 $\quad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$

∵ Invariance $\Rightarrow B(X, [Y, Z]) = B([X, Y], Z)$

$= \alpha(X) B(Y, Z)$ for $X \in \mathfrak{h}$, $Y \in \mathfrak{g}_\alpha$

Cartan's criterion (& invar.) $\exists\, Z \in \mathfrak{g}_{-\alpha}$

$B(Y, Z) \neq 0 \rightsquigarrow$ for this Z $[Y, Z] \neq 0$

(If $T_\alpha \in \mathfrak{h}$ is s.t. $B(T_\alpha, X) = \alpha(X)$ $X \in \mathfrak{h}$

$[Y, Z] = B(Y, Z) T_\alpha$ for $Y \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_{-\alpha}$

Step 2 $\quad \alpha(T_\alpha) \neq 0$

∵ otherwise $\mathfrak{s} = \langle Y, Z, T_\alpha \rangle$ solvable

$\Rightarrow T_\alpha$ nilpot $\underset{\text{diag'ble}}{\Rightarrow} T_\alpha = 0$

Step 3 $\quad \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2(\mathbb{C})$

Step 4