2018 . 10 . 24 no. 1

Summary

- canonical copies of $\mathfrak{sl}_2(\mathbb{C})$ in simple Lie algs
- Killing form

$\mathfrak{g}$ simple Lie alg over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha:\text{root}} \mathfrak{g}_\alpha)$, ...

Proof of (half of) "Fact 2":

$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ contains a subalg isom. to $\mathfrak{sl}_2(\mathbb{C})$.

$\because$ Take $0 \neq Y \in \mathfrak{g}_\alpha$. For $X \in \mathfrak{h}$

Step 1. $B(X, [Y, Z]) \underset{\substack{\text{invariance} \\ \text{\& antisymm.}}}{=} B([X, Y], Z) \underset{Y \in \mathfrak{g}_\alpha}{=} \alpha(X) B(Y, Z)$

Cartan's criterion $\Rightarrow \exists Z$ $B(Y, Z) \neq 0$. Invar. $\Rightarrow Z \in \mathfrak{g}_{-\alpha}$

If $T_\alpha \in \mathfrak{h}$ is the elem s.t. $B(T_\alpha, X) = \alpha(X)$ $(X \in \mathfrak{h})$

we have $[Y, Z] = B(Y, Z) T_\alpha$

Step 2 $\alpha(T_\alpha) \neq 0$

Otherwise $\mathfrak{s} = \langle Y, Z, T_\alpha \rangle$ has T_α in its center $\Rightarrow [\mathcal{D}(\mathfrak{s}), \mathfrak{s}] = 0$, $\mathfrak{s}$ solvable

$\Rightarrow \mathrm{ad}_{T_\alpha}$ on $\mathfrak{g}$ is nilpot. (Lie's thm)

But ad_{T_α} is diag'ble $\Rightarrow \mathrm{ad}_{T_\alpha} = 0$ contr.

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Step 3 $H_\alpha = \frac{2}{\alpha(T_\alpha)} T_\alpha$, $E_\alpha = Y$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ s.t.

$B(E_\alpha, F_\alpha) = \frac{2}{\alpha(T_\alpha)}$ (so $[E_\alpha, F_\alpha] = H_\alpha$)

span a copy of $\mathfrak{sl}_2(\mathbb{C})$. $\because \alpha(H_\alpha) = 2$

Proof of "Fact 1" & "Fact 5".

$\dim \mathfrak{g}_{\pm 1} = 1$, $\dim \mathfrak{g}_{k\alpha} = 0$ $|k| \geq 2$

$\because$ $\mathfrak{s}_\alpha$: copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$

$V = \bigoplus_{k \in \mathbb{C}} \mathfrak{g}_{k\alpha}$ $(\mathfrak{g}_0 = \mathfrak{h})$

Step 1 s_α acts triv. on $\ker(\alpha)$ ($\subset \mathfrak{h} \subset V$)

$\because$ $X \in \ker\alpha \Rightarrow [X, E_\alpha] = \alpha(X) E_\alpha = 0$ ($E_\alpha \in \mathfrak{g}_\alpha$)

$\Rightarrow [E_\alpha, X] = 0$. similarly $[F_\alpha, X] = 0$.

$[H_\alpha, X] = 0$ is from commutivity of $\mathfrak{h}$.

Step 2 "$k\alpha$" in V can only happen for $k \in \frac{1}{2}\mathbb{Z}$

$\because$ ad_{H_α} acts by $2k$ on $\mathfrak{g}_{k\alpha}$.

Step 3 "claim of Fact 5 & 1.

$\because$ 0-eigenspace for $\mathrm{ad}_{H_\alpha} \curvearrowright V$ is exhausted by $\ker(\alpha)$ and 0-wght sp. of $\mathfrak{s}_\alpha$

But $\mathfrak{g}_{k\alpha} \neq 0 \Rightarrow$ will give another 0-eig. vec. (We got Fact 5)

$\mathfrak{g}_\alpha \setminus s_\alpha$ will also create 0-wght sp.

$\Rightarrow \dim \mathfrak{g}_\alpha = 1$ ∎

$\rightsquigarrow$ We also get "Fact 2" ($\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$)

Proof of "Fact 3" $T_\alpha = \dfrac{2}{B(H_\alpha, H_\alpha)} H_\alpha$

$\because$ we already know $H_\alpha = \dfrac{2}{\alpha(T_\alpha)} T_\alpha$ and

$B(T_\alpha, X) = \alpha(X)$

$\Rightarrow B(H_\alpha, H_\alpha) = \dfrac{4}{\alpha(T_\alpha)}$

Proof of "Fact 4" $X \in \mathfrak{h}_0 = \langle H_\alpha : \alpha \text{ root} \rangle_{\mathbb{R}}$

$\Rightarrow B(X, X) \geq 0$, $= 0$ only when $X = 0$

$\because$ $\alpha(H_\beta) \in \mathbb{Z} \Rightarrow (X) \in \mathbb{R}$ for $X \in \mathfrak{h}_0$

$B(X, X) = \sum_{\alpha:\text{root}} \alpha(X)\alpha(X) \geq 0$.

$B(X, X) = 0$ means $\alpha(X) = 0 \ \forall \alpha \Rightarrow X \in \mathfrak{z}(\mathfrak{g}) = 0$

- Killing form

Thm (uniqueness of B) $\mathfrak{g}$: simple W : Weyl grp.

1) $B_{\mathfrak{g}}$ is the unque (up to scalar) invar. bilin. form on $\mathfrak{g}$

2) induced inn. prod. on $\mathfrak{h}^*$ is the unique W-invar. inn. prod.

Proof 1) inv. bilin. form $\longleftrightarrow$ intertwiner $\mathfrak{g} \to \mathfrak{g}^*$

$\mathfrak{g}$ simple $\Rightarrow$ $ad : \mathfrak{g} \curvearrowright \mathfrak{g}$ is irred.

(subrep $\equiv$ ideal)

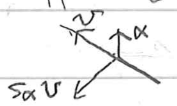
$\rightsquigarrow$ space of bilin. forms is 1-dim.

2) Enough to prove $W \curvearrowright \mathfrak{h}^*$ is irred.

Take $V \subset \mathfrak{h}^*$ W-invar.

Step 1. $\alpha \in \mathfrak{h}^* \setminus V$ root $\Rightarrow$ $\alpha \perp V$.

$\because$ if $\exists v \in V$, $v \not\perp \alpha$, $\Rightarrow$ $\alpha \in V$.



$\alpha \propto v - s_\alpha v \in V$

Step 2 $\mathfrak{g}' = \langle \mathfrak{g}_\alpha : \alpha$ root, $\alpha \in V \rangle_{\text{lin.sp.}}$ $\trianglelefteq_{\text{ideal}} \mathfrak{g}$

$\because$ subalg : $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ $\alpha, \beta \in V \Rightarrow \alpha+\beta \in V$

ideal : take $\beta \perp V$ root, $Y \in \mathfrak{g}_\beta$

$\alpha \in V$ root. $Z \in \mathfrak{g}_\alpha$.

$\Rightarrow$ $\alpha + \beta \notin V$, $\not\perp V$ $\Rightarrow$ $\alpha+\beta$ is not a root.

i.e. $\mathfrak{g}_{\alpha+\beta} = 0$ $\Rightarrow$ $[Y, Z] = 0$

$[Y, H_\alpha] = \underbrace{\beta(H_\alpha)}_{2(\beta,\alpha)/(\beta,\beta)} Y = 0$

$\Rightarrow$ $[Y, \mathfrak{g}'] = 0$. $X \in \mathfrak{h}$ $\Rightarrow$ $[X, \mathfrak{g}'] \subset \mathfrak{g}'$.

Step 3 claim.

$\underset{V=0}{\mathfrak{g}' = 0}$ or $\underset{V = \mathfrak{h}^*}{\mathfrak{g}}$ by Step 2. ☒

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $\quad B_{\mathfrak{g}}(X,Y) = 2n\,\mathrm{Tr}(XY)$

- $\mathrm{Tr}(XY)$ has invariance. i.e.

$$\mathrm{Tr}([Z,X]Y) + \mathrm{Tr}(X[Z,Y]) = 0$$

- $H_1 = E_{11} - E_{22} \in \mathfrak{h}$ $\quad B(H_1, H_1) = 4n$, $\mathrm{Tr}(H_1 H_1) = 2$.

$$\mathrm{ad}_{H_1}(E_{ij}) = \begin{cases} 2E_{12}, & -2E_{21} \\ E_{1j}, & -E_{2j} \quad (j>2) \\ -E_{i1}, & E_{i2} \quad (i>2) \end{cases} \quad \text{otherwise } 0$$

$\rightsquigarrow$ eigenvals	2	-2	1	-1
mult	1	1	$2(n-2)$	$2(n-2)$

$$\Rightarrow \mathrm{Tr}((\mathrm{ad}_{H_1})^2) = 4 + 4 + 2(n-2) + 2(n-2) = 4n$$

Up to normalization, $H_i = E_{ii}$

$B(H_i, H_i) = 2$, $\quad B(H_i, H_{i\pm1}) = -1$, $\quad B(H_i, H_j) = 0$ $\; |i-j|>1$

$$\rightsquigarrow W = \langle (s_i)_{i=1}^{n-1} : s_i^2 = e,\ (s_i s_{i+1})^3 = e,\ s_i s_j = s_j s_i \rangle$$
$$\simeq S_n$$