2018.10.30 no. 1

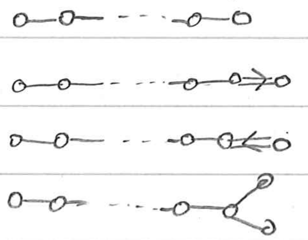
Summary
- Classification of root systems
  - Dynkin diagrams
  - positive roots
  - structure of irred. root system.

- Dynkin diagrams

Goal : classify irred. root system (E, R)

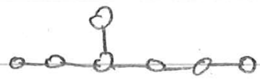
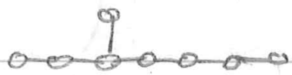
(E : Euclidean sp. $R \subset E$ finite, ----)

by the Dynkin diagrams



- A_n
- B_n
- C_n
- D_n

} n : number of vertices

- E_6  E_7  E_8 
- F_4  , • G_2 

(Rem this will give a classification of simple Lie algs over $\mathbb{C}$

$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C})$, $B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C})$, $C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C})$

$D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C})$)

- Positive roots

$\mathfrak{g}$: simple Lie alg over $\mathbb{C}$, $\mathfrak{h}$: Cartan subalg

$E = \mathbb{R}\Lambda_W$, $R = \{\alpha : \text{root}\}$. $(\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha))$

$E \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{h}^*$, ...

we want say which part of $\mathfrak{g}$ is "uppertriangular"

($\mathfrak{h}$ is "diagonal", ...)

concretely : take $l : E \to \mathbb{R}$ linear which is in "general position" : $\ker l \cap R = \emptyset$.

and call $R^+ = \{\alpha \in R : l(\alpha) > 0\}$ positive roots

$R^- = \{\alpha \in R : l(\alpha) < 0\}$ negative roots

$\alpha \in R^+$ is simple if $\nexists\ \beta, \gamma \in R^+$ s.t. $\alpha = \beta + \gamma$.

Facts 1 any other $l' \rightsquigarrow R^{+'} = w(R^+)$ for $\exists\ w \in W$

2* simple pos. roots form a basis of Λ_R (& E)

(*prove later) 3* $(\alpha, \beta) \le 0$ $(\Rightarrow = -\frac{\sqrt{l}}{2}, l = 0, 1, 2, 3)$

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $\mathfrak{h} = \{X \in \mathfrak{sl}_n(\mathbb{C}) : \text{diag}\}$

$\mathfrak{h}_0 = \langle H_i = E_{ii} - E_{i+1,i+1} : i = 1, \cdots, n-1 \rangle_{\mathbb{R}}$

$E = \mathfrak{h}_0^*$ $\mathfrak{h}_0 \subset \tilde{\mathfrak{h}}_0 = \langle E_{ii} : i = 1, \cdots, n \rangle_{\mathbb{R}}$

$E \xleftarrow{\text{res.}} \tilde{E} = \tilde{\mathfrak{h}}_0^* = \langle L_i : L_i(E_j) = \delta_{ij} \rangle$

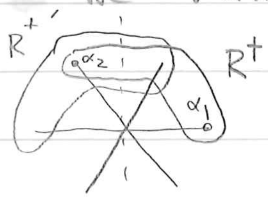
in E, we have $L_1 + \cdots + L_n = 0$ (corr. to trace)

$r_1 > r_2 > \cdots > r_{n-1} > 0 \Rightarrow \begin{cases} l(L_i) = r_i & (i < n) \\ l(L_n) = -(r_1 + \cdots + r_{n-1}) \end{cases}$

$R = \{L_i - L_j : i \neq j\}$ $L_i - L_j \leftrightarrow E_{ij}$

we want l s.t. E_{ij} $(i<j)$ become pos.

Ex.



$s_{\alpha_1}(R^+) = R^{+'}$

generally $|R^{+'} \setminus R^+| = k \Rightarrow \exists\ w = s_{\alpha_1} \cdots s_{\alpha_k}$ s.t. $w(R^+) = R^{+'}$

$L_i - L_{i+1}$ simple pos. root

$\leftrightarrow E_{i,i+1}$ "just above diagonal"

How to draw Dynkin diag.

- Vertices : simple pos. roots $\alpha_1, \cdots, \alpha_n$
- $\frac{(\alpha_i, \alpha_j)}{\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}} = -\frac{\sqrt{l}}{2} \rightsquigarrow l$ edges between α_i & α_j

(Ex. $\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \underset{\alpha_3}{\circ} \Rightarrow \frac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)} = -\frac{1}{2}, (\alpha_1, \alpha_3) = 0$

$l = 2, 3 \rightsquigarrow \underset{\alpha_i}{\circ} \Rightarrow \underset{\alpha_j}{\circ}$ $\underset{\alpha_i}{\circ} \Rrightarrow \underset{\alpha_j}{\circ}$ for $\|\alpha_i\| > \|\alpha_j\|$

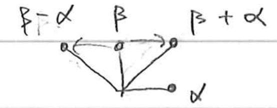
Lem $\alpha, \beta \in R$ $\alpha = \pm \beta$ $p, q = \max$ s.t.

"α-string" $\beta - p\alpha, \beta - p\alpha + \alpha, \cdots, \beta, \beta + \alpha, \cdots, \beta + q\alpha$

are roots. Then $p + q \leq 3$, $p - q = 2\dfrac{(\alpha, \beta)}{(\alpha, \alpha)}$ $(\in \mathbb{Z})$

$(n_{\beta,\alpha})$

B_2 :



Proof. $s_\alpha \in W$ reflects α-string. $\beta + q\alpha \longleftrightarrow \beta - p\alpha$.

Concretely $s_\alpha(\gamma) = \gamma - 2\dfrac{(\alpha, \gamma)}{(\alpha, \alpha)}\alpha$

so $s_\alpha(\beta + k\alpha) = \beta - (n_{\beta,\alpha} + k)\alpha$ $\quad p = n_{\beta,\alpha} + q$

With $\beta' = \beta - p\alpha$: $3 \geq n_{\beta',\alpha} = p + q$

Prop $\alpha, \beta \in R$, $\alpha = \pm \beta$

1) $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta$ is a root

$(\alpha, \beta) < 0 \Rightarrow \alpha + \beta$ is a root

2) $(\alpha, \beta) = 0$ either $\left(\begin{array}{l} \alpha \pm \beta \text{ are both roots} \\ \sim \text{ are both not roots} \end{array}\right.$

Proof 1) γ root $\Leftrightarrow -\gamma = s_\gamma(\gamma)$ root

enough to show $(\alpha, \beta) > 0 \Rightarrow \beta - \alpha$ root.

But $s_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$. $n_{\beta,\alpha} > 0$

"α-string" through β contains $\beta - \alpha$.

2) From Lem. $(\alpha, \beta) = 0$ means $p = q$.

$p = q = 0 \Rightarrow \beta \pm \alpha$ are not roots

$p = q > 0 \Rightarrow \beta \pm \alpha$ are roots. ∎

Cor α, β simple pos.

1) $\alpha - \beta$ ($\&$ $\beta - \alpha$) not root

2) $(\alpha, \beta) \leq 0$

Proof 1) from def of simplicity

$\because$ $\alpha - \beta \in R \Rightarrow$ either $\alpha - \beta$ or $\beta - \alpha$ is pos

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2) Prop 1) and Cor 1).

Lem.. simple pos. are lin. indep.

$\because$ They $\begin{cases} \text{belong to } \ell(\gamma) > 0 \\ \text{have non-acute angles} \end{cases}$

Cor. simple pos. roots form a basis of E.

$\because$ gen. $\mathbb{N} \cdot R^+$ as semigrp.

$R = R^+ \cup (-R^+)$