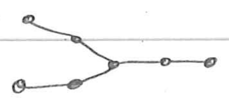
2018.10.31 no.1

Summary
- classification of Dynkin diag.
- Dynkin diag. to simple Lie alg.

∘ Class. Dynkin.
(E, R) root system $R = R^+ \amalg R^-$ choice of pos. roots.
$\rightsquigarrow$ $\Pi = \{\alpha_i : i = 1, \dots, n\}$ simple pos rts
Dynkin diag. Γ vertex set $= \Pi$
Goal. we want to say Γ is one of A_n, B_n, C_n, D_n, E_k $(6 \le k \le 8)$, F_4, G_2

Put $e_i = \frac{1}{\sqrt{(\alpha_i, \alpha_i)}} \alpha_i$ (rescaled unit vec)
$\rightsquigarrow$ $(e_i, e_j) = -\frac{\sqrt{d}}{2}$ $d = 0, 1, 2, 3$.

Rem. $4(e_i, e_j)^2 = d$ = number of edges between α_i and α_j in Γ
Not. $i \sim j$ for $\exists$ edge α_i & α_j
We'll check Γ
P1 does not have loops
P2 any vertex is connected to < 4 edges
P3 does not contain



("the rest" are similar, FH §21.2)
P1) $J \subset \Pi$, $|J| = k \Rightarrow \exists$ at most $k-1$ (unord.) pairs (j, j') with $j, j' \in J$ & $e_j, e_{j'}$ are conn.
$v = \sum_j e_j \Rightarrow (v, v) = \sum_j (e_j, e_j) + \sum_{j \sim j'} (e_j, e_{j'})$
$\le k + 2 \times (\#(j, j') \text{ as abv}) \times (-\frac{1}{2})$

this has to be pos. $\square$

P2) fix i. Want: $\sum_j \underbrace{\#(\text{edges between } \alpha_i \,\&\, \alpha_j)}_{4(e_i,e_j)^2 \text{ by Rem.}} < 4$

Step 1. $i \sim j$ & $i \sim j' \Rightarrow j \not\sim j'$

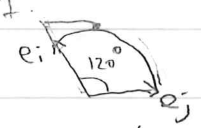
$\because$ P1

Step 2. estim. of claim.

$(e_j : i \sim j)$ are mut. orth, don't span e_i

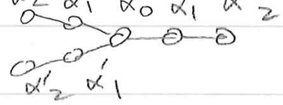
$\Rightarrow \sum (e_i, e_j)^2 < \|e_i\| = 1$ $\square$

P3) S1.



$\frac{2e_i + e_j}{\sqrt{3}}$ is a unit vec.

Step 2 contra. from



$v = \frac{2e_1 + e_2}{\sqrt{3}}$, v', v'' unit vecs.

• $(e_0, v) = \frac{2}{\sqrt{3}}(e_0, e_1) = -\frac{1}{\sqrt{3}}$

v, v', v'' mut orth, don't span e_0

$\Rightarrow \underbrace{(e_0, v)^2}_{\frac{1}{3}} + \underbrace{(e_0, v')^2}_{\frac{1}{3}} + \underbrace{(e_0, v'')^2}_{\frac{1}{3}} < \underbrace{(e_0, e_0)}_{1.}$ $\square$

Dynkin to $\mathfrak{g}$

Cartan matrix. $A = (a_{ij})_{i=1}^{n}$ $(n = \#\Pi)$

$a_{ij} = n_{\alpha_i, \alpha_j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}$ (another conv. n_{α_j, α_i})

Rem $a_{ii} = 2$, $i \neq j \Rightarrow a_{ij} \in \{0, -1, -2, -3\}$

$A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B_2 : \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$, $G_3 : \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

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Goal: $\mathfrak{g}$ cplx simple Lie, $\mathfrak{h} \subset \mathfrak{g}$ Cartan

R roots, $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in R^+$

$(R = R^+ \amalg R^-, \Pi)$ $H_\alpha \in \mathfrak{h}$ so $\langle E_\alpha, F_\alpha, H_\alpha \rangle \simeq \langle E, F, H \rangle$

Goal describe structure of $\mathfrak{g}$ interms of

- $E_\alpha, F_\alpha, H_\alpha$ for $\alpha \in \Pi$ as generators
- relation in terms of Cartan mat A

  (can read off from Dynkin diag)

$\rightsquigarrow$ which will give

- <u>existence</u> of $\mathfrak{g}$ for given Dynkin diag
- <u>uniqueness</u> $(\mathfrak{g}_i, \mathfrak{h}_i)$ $i = 1,2$ gave

  same Γ $\Rightarrow$ $\exists$ iso φ $\mathfrak{g}_1 \to \mathfrak{g}_2$

  s.t. $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$

(we'll remove the choice of $\mathfrak{h}$ later)

"Obvious" relations

- $[E_\alpha, F_\alpha] = H_\alpha$ by def
- $[E_\alpha, F_\beta] = 0$ (would give elem in $\mathfrak{g}_{\alpha-\beta}$

  but $\alpha - \beta$ is not a root)
- $[H_\alpha, H_\beta] = 0$
- $[H_{\alpha_i}, E_{\alpha_j}] = a_{ji} E_{\alpha_j}$, $[H_{\alpha_i}, F_{\alpha_j}] = -a_{ij} F_{\alpha_j}$

$\because$ equiv. to $a_{ji} = \alpha_j(H_{\alpha_i})$

$$\beta(H_{\alpha_i}) = \frac{2B(\beta, \alpha_i)}{B(\alpha_i, \alpha_i)}$$ for inv. bilin. form B

(10.23)

Not-so-obvious relation (Serre relation)

$$\mathrm{Ad}_{E_{\alpha_i}}^{1-a_{ji}}(E_{\alpha_j}) = 0, \quad \mathrm{Ad}_{F_{\alpha_i}}^{1-a_{ji}}(F_{\alpha_j}) = 0 \quad (i \neq j)$$

Ex. $\mathfrak{sl}_3(\mathbb{C})$ claim is $[E_{12}, [E_{12}, E_{23}]] = 0$

$[E_{21}, [E_{21}, E_{32}]] = 0$

Pf of Serre rel. for E's.

look at α_i - string through α_j

$\alpha_j - \alpha_i$ not root

$\rightsquigarrow \alpha_j, \alpha_j + \alpha_i, \cdots, \alpha_j + (\underbrace{-n_{\alpha_j,\alpha_i}}_{-a_{ji}})\alpha_i$