Summary
- Construction of simple Lie algs from root sys.
  - free Lie algs
  - ideal from Serre's rel.
  - implementing Weyl group action

◦ Free Lie algs

Want : construct Lie algs from "generators & rels"

$K$ : comm. field

Case 1 No rels.

$S$ : set (of "generators")

$\rightsquigarrow M(S) = \coprod_{k=1}^{\infty} S_k$, $S_1 = S$, $S_n = \coprod_{p=1}^{n-1} S_p \times S_{n-p}$

"set of "words" with letters in $S$ $a, ab, (ab)c, \dots$

distinguish parenthesization. $(ab)c \neq a(bc)$

$L_K(S) = \underline{K[M(S)]} / \langle aa, (ab)c + (bc)a + (ca)b \rangle$

formal lin comb. of $M(S)$ $a, b, c \in M(S)$

nonassoc. alg

$x_a$ : img of $a$ in $L_K(S)$.

Lie bracket by $[x_a, x_b] = x_{ab}$

$L_K(S)$ : free Lie alg generated by $S$

Case 2. Put relations

$R$ : set of "relations" $\subset K[M(S)]$

(like $a \cdot b - c \leftrightsquigarrow$ "$[x_a, x_b] = x_c$")

$L_K(S; R) = L_K(S) / \langle$ img of $r \in R \rangle$

Lie alg gen. by $S$, with rel $R$.

(or, by $(x_t)_{t \in S}$ )

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- Ideal from Serre's rel.

$(E, R)$ root sys, $R = R^+ \amalg R^-$ pos - neg. ...

$\Pi = \{\alpha_1, \cdots, \alpha_n\}$ simple pos. roots.

$a_{ij} = n_{\alpha_i \alpha_j} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, $A = (a_{ij})_{i,j=1}^n$ Cartan mat

$\tilde{\mathfrak{g}}$ : Lie alg with

generators : $e_i, f_i, h_i$ $i = 1, \cdots, n$

$(S = \Pi \times \{-1, 0, 1\}$, $x(i, 1) = e_i$, $x(i, -1) = f_i$, $x(i, 0) = h_i)$

relations $\cdot$ $[h_i, h_j] = 0$

$\cdot$ $[e_i, f_i] = h_i$ $\cdot$ $[e_i, f_j] = 0$ $(i \neq j)$

$\cdot$ $[h_i, e_j] = a_{ji} e_j$, $[h_i, f_j] = -a_{ji} f_j$

( all rels in the corresp. semisimple Lie alg <u>except</u> for the Serre rels )

$\tilde{\mathfrak{g}}_\pm \rightarrow$

Rem. $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(h_i) = -h_i$

is an aut. of $\tilde{\mathfrak{g}}$ ("$\theta(X) = -X^t$")

$x_{ij} = \mathrm{ad}_{e_i}^{1-a_{ji}}(e_j)$, $b_+$ = ideal of $\mathfrak{g}_+$ gen'ed by $(x_{ij})_{ij}$

$y_{ij} = \mathrm{ad}_{f_i}^{1-a_{ji}}(f_j)$, $b_- = \cdots\cdots \mathfrak{g}_-$

Prop. $[\tilde{\mathfrak{g}}_-, x_{ij}] = 0$ $([\tilde{\mathfrak{g}}_+, y_{ij}] = 0)$

Cor. $b_\pm \triangleleft \tilde{\mathfrak{g}}$ $(b_- = \theta(b_+))$

$\because$ $\cdot$ $h_i$ stabilize $b_+$. ( $[e_{k_1}, [\cdots, [e_{k_p}, x_{ij}] \cdots]$ eigenvec. of $\mathrm{ad}_{h_i}$ )

$\cdot$ $e_i$ obviously stab. $b_+$

$\cdot$ Prop $\Rightarrow$ $f_i$ also stab $b_+$

Proof of Prop.

Step 1. $[f_k, x_{ij}] = 0$ for $k \notin \{i, j\}$.

$\because$ $[f_k, e_i] = 0 = [f_k, e_j]$

Step 2 $[f_j, x_{ij}] = 0$

$\because$ $f_j$ comm. with $e_i$ $\Rightarrow$ $\mathrm{ad}_{f_j}$ comm. with $\mathrm{ad}_{e_i}$

$\Rightarrow [f_j, x_{ij}] = \mathrm{ad}_{e_i}^{1-a_{ji}} \underbrace{\mathrm{ad}_{f_j}(e_j)}_{-h_j}$

$= \mathrm{ad}_{e_i}^{-a_{ji}} (\underbrace{[e_i, h_j]}_{-a_{ij} e_i})$

$a_{ij} = 0 \Rightarrow$ this is zero

$a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0 \Rightarrow$ we'll have $\mathrm{ad}_{e_i}(e_i) = 0$

Step 3 $[f_i, x_{ij}] = 0$

$\because$ use $[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}] = -\mathrm{ad}_{h_i}$, $[\mathrm{ad}_{h_i}, \mathrm{ad}_{e_i}] = 2\,\mathrm{ad}_{e_i}$

$\rightsquigarrow [\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}^k] = -k\, \mathrm{ad}_{e_i}^{k-1} (\mathrm{ad}_{h_i} + k - 1)$

(e.g. $[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}^2] = \underbrace{[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}]\mathrm{ad}_{e_i}}_{-\mathrm{ad}_{e_i}\mathrm{ad}_{h_i} + 2\,\mathrm{ad}_{e_i}} + \mathrm{ad}_{e_i}[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}]$

Use this for $k = 1 - a_{ji}$, & $(\mathrm{ad}_{h_i} - a_{ji}) e_j = 0$ ☒

Goal: $\mathfrak{g}_A = \tilde{\mathfrak{g}} / (\mathfrak{b}_+ + \mathfrak{b}_-)$ is semisimple.

$E_i$ = img of $e_i$, $F_i$, $H_i$ similar.

∘ Implementing Weyl grp.

Want: impl. $W \curvearrowright \mathfrak{h}$ as "Adjoint by elems of $G$"

<u>Derivation</u> $D: \mathfrak{g} \to \mathfrak{g}$, $D([x, y]) = [x, D(y)] + [D(x), y]$

Ex. $D(x) = [z, x] = \mathrm{ad}_z(x)$

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If $D$ is (locally) nilpotent $\forall x \in \mathfrak{g}\ \exists k\ D^k(x)=0$

$e^D : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} D^k(x)$ makes sense (Char $K=0$)

Lem $e^D$ is Lie alg aut.

$\because$ By ind. $\frac{1}{k!} D^k([x,y]) = \sum_{n=0}^{k} \frac{1}{n!(k-n)!} [D^n(x), D^{k-n}(y)]$

For $\mathfrak{g} = \mathfrak{g}_A$ from before

$ad_{E_i}$, $ad_{F_i}$ are (locally) nilpot.

$\theta_i = e^{ad_{E_i}} e^{-ad_{F_i}} e^{ad_{E_i}} \in Aut(\mathfrak{g})$.

Rem $e^{ad_X}(Y) = e^X Y e^{-X}$ for matrices

$e^{ad_E} e^{-ad_F} e^{ad_E}(Y) = Ad(\underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{\begin{bmatrix}0&1\\-1&0\end{bmatrix}})(Y)$ $Y \in M_2(K)$

Lem $\theta_i(H_j) = H_j - a_{ij} H_i$

($\rightsquigarrow \theta_i$ induces $s_{\alpha_i} \curvearrowright \mathfrak{h}^*$, $\mathfrak{h} = \text{span of } (H_i)_i$)

$\because$ Direct computation.

- semi-simplicity of $\mathfrak{g}_A$

Th'm $(E,R)$ (irred) root system

$\Rightarrow \mathfrak{g}_A$ (semi) simple.

Proof. Step 1 Enough to prove $\nexists$ nonzero comm. ideal.

$\because$ $Rad(\mathfrak{g}) \neq 0 \Rightarrow \exists\ 0 \neq \mathfrak{b} \triangleleft \mathfrak{g}$, comm. (09. 26)

Step 2 $\mathfrak{b} \triangleleft \mathfrak{g} \Rightarrow \mathfrak{b} = \mathfrak{b} \cap \mathfrak{h} \oplus (\bigoplus_{\alpha : \text{roots}} \mathfrak{b} \cap \mathfrak{g}_\alpha)$

$\because$ From $ad_{h_i}$ $\left( \begin{array}{l} \text{mutually commute} \\ \text{diag'ble} \end{array} \right.$

$\mathfrak{g} = (\bigoplus_\alpha \mathfrak{g}_\alpha) \oplus \mathfrak{h}$ (fact 1)

Step 3. $\mathfrak{b} \triangleleft \mathfrak{g}$, $\mathfrak{b}$ comm $\Rightarrow$ $\mathfrak{b} \cap \mathfrak{g}_{\alpha_i} = 0$.

$\because$ $\dim \mathfrak{g}_{\alpha_i} = 1$ & $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] \neq 0$.

(Fact 2)

$\exists\, w \in W$ s.t. $w(\alpha_i) = -\alpha_i$ $\quad (w = s_{\alpha_i})$

$\mathfrak{b} \triangleleft \mathfrak{g}$, $\mathfrak{b} \cap \mathfrak{g}_{\alpha_i} \neq 0 \Rightarrow \mathfrak{b} \cap \mathfrak{g}_{-\alpha_i} \neq 0$

$\because$ action of $w = e^{\mathrm{Ad}x_1} \cdots e^{\mathrm{Ad}x_k}$.

But this will imply $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{b}$.

Step 4. $\mathfrak{b} \cap \mathfrak{h} \neq 0 \Rightarrow \exists\, i$ s.t. $E_i \in \mathfrak{b}$

($\Rightarrow \mathfrak{b}$ noncomm.)

$\because$ For simplicity suppose $(E, R)$ irred

$\rightsquigarrow$ $W \curvearrowright E$ is irreducible (cf. 10.24 Thm)

$\mathfrak{b} \cap \mathfrak{h}$ is $W$-invar $\Rightarrow \mathfrak{b} \cap \mathfrak{h} = \mathfrak{h}$.

then $2E_i = [H_i, E_i] \in \mathfrak{b}$. $\square$

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Complements

Fact 1 : $\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\mu \in \Lambda_R} \mathfrak{g}_\mu \right)$ by const.

$\Lambda_R$ : root lattice

$\rightsquigarrow$ want : $\mathfrak{g}_\mu = 0$ for $\mu \in \Lambda_R \setminus R$.

Step 1 $\mu$ not mult. of root

$\because$ $\exists\, w \in W$ s.t. $w\mu \notin \mathbb{N}R \cup (-\mathbb{N}R)$

$\Rightarrow$ $\tilde{\mathfrak{g}}_{w\mu} = 0$ by const. $\Rightarrow$ $\mathfrak{g}_{w\mu} = 0$

$\Rightarrow$ $\mathfrak{g}_\mu = 0$

Step 2 $\mu = m\alpha$ $\quad m > 2$, $\alpha \in R$.

$\because$ $\tilde{\mathfrak{g}}_\mu = 0$ bc. $\tilde{\mathfrak{g}}_+ \simeq L_K((e_i)_{i=1}^n)$

Fact 2 : $\tilde{\mathfrak{g}}_+ = L_K((e_i)_{i=1}^n) \Rightarrow \dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$

If $e_i \in \mathfrak{b}_+$ $\quad h_i = [e_i, f_i] \in \mathfrak{b}_+$ not possible. $\square$