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Summary
- Uniqueness of Cartan subalg.
- Borel subgroups, (full) flag manifold
- compact form

- Uniqueness of Cartan subalg

$\mathfrak{g}$ complex (semi) simple Lie alg (like $\mathfrak{sl}_n(\mathbb{C})$)

Th'm $\mathfrak{h}, \mathfrak{h}'$ Cartan subalg
$\rightsquigarrow \exists g \in G$ s.t. $Ad_g(\mathfrak{h}) = \mathfrak{h}'$.

G : complex (semi) simple Lie grp corr. to $\mathfrak{g}$
(like $SL_n(\mathbb{C})$)

Conseq. choice of a Cartan subalg $\mathfrak{h} \subset \mathfrak{g}$ is essentially unique :
- root system of $(\mathfrak{g}, \mathfrak{h})$ does not depend on this choice
- Dynkin diagrams classify $\mathfrak{g}$, not just $(\mathfrak{g}, \mathfrak{h})$
(having $\mathfrak{g} \simeq \mathfrak{g}'$ is "same" as $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}', \mathfrak{h}')$)

Outline for the proof of th'm.

Step 1. Find $Y \in \mathfrak{h}$ or $h \in H$ (subgrp of G corr. to $\mathfrak{h}$) s.t.
$$\mathfrak{h} = \begin{cases} \mathrm{Cent}_{\mathfrak{g}}(Y) = \{X : [X, Y] = 0\} \\ \mathrm{Cent}_{\mathfrak{g}}(h) = \{X : Ad_g(X) = X\} \end{cases}$$

$\because$ Y, h : any elem in "general position"
$$\begin{cases} \alpha(Y) \neq 0 \text{ for any root } \alpha. \ (Y \text{ is regular}) \\ h \text{ generates a dense subgroup of } H \end{cases}$$
(like $\begin{bmatrix} e^{2\pi\sqrt{-1}\theta_1} & & \\ & \ddots & \\ & & e^{2\pi\sqrt{-1}\theta_n} \end{bmatrix}$ $\theta_1, \cdots, \theta_{n-1}$ lin. indep. $/\mathbb{R}$)

Step 2 Find $g \in G$ s.t. $Ad_g(Y) \in \mathfrak{h}'$ or $Ad_g(h) \in H'$
$\rightsquigarrow \mathfrak{h}' = Ad_g(\mathfrak{h})$

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To carry out Step 2 (by Borel subgrp)

- Use quotient G/B' (full flag manifold) by a Borel subgroup, & Lefschetz fixed pt thm for action of h. $\rightsquigarrow$ $hg^{-1}B' = g^{-1}B'$ for some g^{-1}
  $SL_2(\mathbb{C}) \rightsquigarrow SL_2(\mathbb{C}) / \{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \} \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$.
- Use compact form $K \subset G$ & prove uniqueness (up to conjugation) for $T = K \cap H$
  - any elem. of K is conj. to an elem. of T.
- $\mathfrak{h}_{reg} = \{ Y : \text{regular} \in \mathfrak{h} \}$
  $F : G \times \mathfrak{h}_{reg} \to \mathfrak{g}$, $(g, Y) \mapsto Ad_g(Y)$
  img is Zariski open (..
  $\rightsquigarrow$ $Ad_{g_0}(Y) = Ad_{g_0'}(Y')$ for some (g_0, Y) & (g_0', Y')
  $\rightsquigarrow$ $g = g_0'^{-1} g_0$ conjugates Y into $\mathfrak{h}'$.
- Borel subgroups
  $\mathfrak{g}$: (semi) simple $/\mathbb{C}$, $\mathfrak{h} \subset \mathfrak{g}$ Cartan
  $R = R_+ \amalg R_-$ decomp of roots,
  ($\Pi = \{\alpha_1, \cdots, \alpha_n\}$ simple pos. roots)
  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ $\quad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$
  $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. Borel subalgs ("upper/lower triang")

Prop. $\mathfrak{b}_\pm$ are maximal solvable subalgs containing $\mathfrak{h}$.

Pf. $\mathfrak{h} \subset \mathfrak{g}' \subset \mathfrak{g}$ subalgs $\Rightarrow$ $\mathfrak{g}' = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}' \cap \mathfrak{g}_\alpha)$
if $\mathfrak{b}_+ \subsetneq \mathfrak{g}'$ $\quad \exists\, \alpha \in R_-$ s.t. $\mathfrak{g}' \cap \mathfrak{g}_\alpha \neq 0$
$\mathfrak{g}'$ will contain $\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$ notsolvable ☐

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Borel subalg : max. solvable subalg of $\mathfrak{g}$.
(all Borel subalgs are conjugate under $G \overset{Ad}{\curvearrowright} \mathfrak{g}$).

We'll look at B = subgroup of G for $\mathfrak{b}_+$.

$$SL_n(\mathbb{C}) \rightsquigarrow B = \left\{ \begin{bmatrix} * & * & \cdots & * \\ & * & & \vdots \\ 0 & & \ddots & \\ & & & * \end{bmatrix} \right\}.$$

G/B : full flagmanifold.

Facts : • $G/B = \coprod_{w \in W} B[w]$. $\left(G = \coprod_{w \in W} BwB \right)$
Bruhat decomposition.

each $B[w] \simeq \mathbb{C}^{\ell(w)}$ for "length" func.

$$\ell : W \to \mathbb{N}.$$

(• G/B is a projective variety).

$$\rightsquigarrow H^n(G/B; \mathbb{Q}) = \begin{cases} \mathbb{Q}^{\oplus \#\{w : 2\ell(w) = n\}} & n : \text{even} \\ 0 & n : \text{odd}. \end{cases}$$

Lefschetz fixed pt formula.

X : "nice" cpt. top. sp. (finite CW cplx. ...)

$f : X \to X$ ct map.

$$\rightsquigarrow \#\{p \in X : f(p) = p\} \geq \sum (-1)^n \operatorname{Tr}\left(f_* |_{H^n(X;\mathbb{Q})}\right) =: \Lambda_f$$

Cor. $\forall h \in G$ $\exists g \in G$ s.t. $hg^{-1}B = g^{-1}B$.

$\because$ G/B is "nice", $h \curvearrowright G/B$ is homotop. to id. $\Rightarrow \Lambda_h = \sum \dim H^{2n}(X; \mathbb{Q}) = |W|$.

Application to uniqueness

$\mathfrak{h}, \mathfrak{h}'$ Cartan, $h \in H$ s.t. $\mathfrak{h} = \operatorname{Cent}_{\mathfrak{g}}(h)$

look at $h \curvearrowright G/B'$, find g as above.

$\rightsquigarrow Ad_g(h) \in B' \rightsquigarrow$ by semisimple - nilpot decomp
$Ad_g(h) \in H'$

- Compact form.

Want : real Lie subalg $\mathfrak{g}_c \subset \mathfrak{g}$ s.t.

- $G_c \subset G$ is compact (small).
- $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$. (but recovers $\mathfrak{g}$)

How to do : if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we want

$\mathfrak{g}_c = \mathfrak{su}(n) = \{ X \in M_n(\mathbb{C}) : -\bar{X}^t = X \}$

$\sigma : X \mapsto -\bar{X}^t$ is conjug. lin. autmorphism

s.t. $\sigma^2 = \mathrm{Id}$.

(and root vec $E_{ij} \mapsto -E_{ji}$)

Generally : $(\mathfrak{g}, \mathfrak{h})$, $R = R_+ \sqcup R_-$

$E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ $H_\alpha = [E_\alpha, F_\alpha]$ for $\alpha \in R_+$

$\rightsquigarrow$ $\sigma(E_\alpha) = -F_\alpha$, $\sigma(F_\alpha) = -E_\alpha$, $\sigma(H_\alpha) = -H_\alpha$.

(enough to define on simple pos. roots)

and extend as conjugate linear.

Prop. 1) σ is an aut. of $\mathfrak{g}$

2) on $\mathfrak{g}_c = \{ X \in \mathfrak{g} : \sigma(X) = X \}$, the Killing form is negative definite

3) $G_c < G$ for $\mathfrak{g}_c$ is compact

4) $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$.

Proof 2) $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$. is $\sqrt{-1}\, \mathfrak{h}_0$ ($\mathfrak{h}_0 = \langle H_\alpha \rangle_{\text{real}}$.)

$\rightsquigarrow$ $\forall \alpha \in R$ $\alpha : \mathfrak{h}_c \to \sqrt{-1}\,\mathbb{R}$.

$\mathfrak{g}_c$ has basis $F_\alpha - E_\alpha$, $\sqrt{-1}(E_\alpha + F_\alpha)$ $(\alpha \in R_+)$

and $\sqrt{-1}\, H_\alpha$ $(\alpha \in \Pi)$

these are orth., $B(X, X) < 0$ by direct comp.

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$\left(F - E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sqrt{-1}(E+F) = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix} \right).$

Use

- $B_{\mathfrak{g}}(E_\alpha, F_\alpha) > 0$, $B(E_\alpha, E_\alpha) = 0$, etc.
- $B_{\mathfrak{g}}(\sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha) = -\sum \beta(H_\alpha)^2$

3) $G_c \curvearrowright \mathfrak{g}_c$ img is in $O(\mathfrak{g}_c; B)$ cpt. grp. $(\cong O(\mathbb{R}^{\dim \mathfrak{g}_c}))$

kernel : $\{g \in G_c : Ad_g = Id\} \subset \{g \in G : Ad_g = Id\}$ finite.