2018.11.13. no.1

Summary
- compact form, cont'd
  - uniqueness of maximal torus
- highest weight theory

◦ Compact form, continued
(Recall cpt invol & do Prop from last wk)

Rem. Good & bad of cpt frm
G : any $X \in \mathfrak{g}_c$ is semisimple (in $\mathfrak{g} = \mathfrak{g}_c \otimes \mathbb{C}$)
(cf. any $g \in SU(n)$ is diagonalizable)
B : we cannot take root (or weight) decomp. over $\mathbb{R}$. ($X \in \mathfrak{g}_c \rightsquigarrow \pi(X)$ have eigenvals λ in $\sqrt{-1}\mathbb{R}$; $e^{\lambda} \in \mathbb{T} = \{z : |z| = 1\}$

Fact $\mathfrak{g}_c$ is unique up to conjugation.
i.e. $\mathfrak{g}_1 < \mathfrak{g}$ real Lie subalg generates a maximal cpt subgroup $G_1 < G$
$\Rightarrow \exists g \in G$ s.t. $Ad_g(\mathfrak{g}_1) = \mathfrak{g}_c$ (so $g G_1 g^{-1} = G_c$)

Notn $K = G_c$ $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$, $T < G$ corr. to $\mathfrak{h}_c$
$\Rightarrow$ $\mathfrak{h}_c$ is a maximal comm. subalg of $\mathfrak{g}_c$
T is a maximal comm. subgrp of K
(<u>maximal torus</u>)
(Rem about $\Lambda_W = \hat{T}$.)

Th'm T' max. comm. $< K \Rightarrow \exists g \in K$ s.t. $gTg^{-1} = K'$
$\mathfrak{h}'_c$ ~ $< \mathfrak{g}_c \Rightarrow$ ~ $g \mathfrak{h}_c g^{-1} = \mathfrak{h}'_c$

Proof we'll do $gTg^{-1} = T'$ (but the proof does $Ad_g(\mathfrak{h}_c) = \mathfrak{h}'_c$)
Step 1 $\exists h$ s.t. $T = Cent_K(h)$.

skip?

$\because$ take $h \in T \simeq \mathbb{R}^n/\mathbb{Z}^n$ s.t. $\{h^k : k \in \mathbb{Z}\}$ is dense in T

$\rightsquigarrow$ enough to find $g \in K$ s.t. $ghg^{-1} \in T'$.

Step 2 $\forall h \in K\ \exists g \in K$ s.t. $ghg^{-1} \in T'$

Step 2-1 $\mathfrak{g}_c \to K$, $X \mapsto e^X$ is surj.

Outline • K cpt $\underset{\text{Hopf-Rinow}}{\Rightarrow}$ $\forall h \in K$ $\exists$ geodesic from e to h (for biinv. metric)

• geod. from e $\equiv$ exponential curve $(e^{tX})_t$ (for biinv. met)

Step 2-2 $\forall X \in \mathfrak{g}_c\ \exists g \in K$ s.t. $\mathrm{Ad}_g(X) \in \mathfrak{h}'_c$

(then $e^X = h \Rightarrow \mathrm{Ad}_g(h) \in T'$)

$\because$ (Y, Z) invariant inn. prod on $\mathfrak{g}_c$ $(-B_{\mathfrak{g}}(Y, Z))$

Take $Y' \in \mathfrak{h}'_c$ s.t. $\mathrm{Cent}_{\mathfrak{g}_c}(Y') = \mathfrak{h}'_c$.

(enough to take Y' s.t. $(e^{tY'})_t$ dense in T')

Put $A_X = \{\mathrm{Ad}_g(X) : g \in K\}$ cpt subset $\subset \mathfrak{g}_c$

$\rightsquigarrow$ $\exists X' \in A_X$ s.t. $X'' \mapsto |X'' - Y'|$ is min. at X'

Claim : $X' \in \mathfrak{h}'_c$.

$\because$ $|\mathrm{Ad}_{e^{tZ}}(X') - Y'|^2$ is smallest at $t=0$

$\Rightarrow$ deriv. is 0 at $t=0$

$\Rightarrow$ $(X' - Y', [Z, X']) = 0$ for all $Z \in \mathfrak{g}_c$

invar. for $\mathrm{ad}_{X'}$ $\Rightarrow$ $(\mathrm{ad}_{X'}(Y'), Z) = 0$

i.e. $\mathrm{ad}_{X'}(Y') = 0$ $\Rightarrow$ $X' \in \mathrm{Cent}_{\mathfrak{g}_c}(Y')$. $\square$

Rem. weight lattice Λ_W = Pontryagin dual of T

$= \{\varphi : T \to \mathbb{C}^\times \text{ cont. hom}\}$

(1-dim rep. of T)

- Highest weight theory.

$\mathfrak{g}$ (semi) simple $\mathfrak{h}$, Λ_W, $\cdots$

(π, V) fin. dim rep $\rightsquigarrow$ $V = \bigoplus_{\omega \in \Lambda_W} V_\omega$.

$\underset{\text{weight sp.}}{V_\omega} = \{ v \in V \ ; \ \pi_X v = \omega(X) v \ \ (X \in \mathfrak{h}) \}$

Want
- find "highest weight" of V.
- highest weight determines irred. rep

Fix pos / neg. dec. of roots $R = R_+ \amalg R_-$

$\rightsquigarrow$ simple pos. roots $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset R_+$

Def. • fundamental weights $\omega_1, \cdots, \omega_n$:

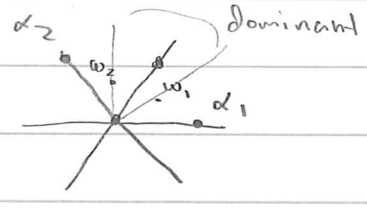
$\omega_i(H_{\alpha_j}) = \delta_{i,j}$. $\Leftrightarrow$ $\dfrac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{i,j}$.

(10/23) typo.

$\rightsquigarrow$ basis of Λ_W.

- $\omega \in \Lambda_W$ is dominant if $\omega(H_{\alpha_j}) \geq 0 \ \forall j$

i.e. $\omega = \sum m_i \omega_i$ $m_i \geq 0$ $\Updownarrow$ $\omega(R_+) \geq 0$

A_2 

$P_{++} = \{ \omega : \text{dominant} \}$.

Fact. $W \curvearrowright \Lambda_W$ P_{++} is a fundamental dom.
(any W-orbit intersects w/ P_{++} exactly once)

Recall $\pi(E_{\alpha_i})$ nilpot $\Rightarrow$ $\exists$ highest weight vec.

i.e. $0 \neq v \in V_\omega \cap \left(\bigcap_{i=1}^{n} \operatorname{Ker} \pi(E_{\alpha_i}) \right)$ for some ω.

Th'm. $\omega \in P_{++}$ $\Rightarrow$ $\exists !$ irred. rep. (π, V) with highest wght. ω.

Ingredient. : Verma module $V(\omega)$
- infin dim $\mathfrak{g}$-module, dist. vec. $v_\omega \in V(\omega)$

2018.11.13. no4

- (π', V') has highest wght vec $v' \in V'_\omega$
  $\Rightarrow \exists!$ $\mathfrak{g}$-hom $V(\omega) \to V'$, $v_\omega \mapsto v'$
- $\exists!$ nontriv. max. $\mathfrak{g}$-inv. subsp. $M_\omega < V(\omega)$
  i.e. $V(\omega)$ has unique irred. quot. V^ω

Construction.

$U(\mathfrak{g})$, $U(\mathfrak{b}_+)$, $U(\mathfrak{h})$ univ. env. algs.

- $U(\mathfrak{b}_+) \subset U(\mathfrak{g})$
- $\mathfrak{b}_+ \to \mathfrak{h}$ prj. is Lie alg hom
  $\Rightarrow U(\mathfrak{b}_+) \to U(\mathfrak{h})$
- ω induces $U(\mathfrak{h}) \xrightarrow{\psi_\omega} \mathbb{C}$. 1-dim rep

$\rightsquigarrow V(\omega) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} {}_{\psi_\omega}\mathbb{C}$, $v_\omega = 1 \otimes 1$

any <u>proper</u> submodule doesn't contain v_ω.

$\Rightarrow M_\omega = \bigcup_{\substack{W \subset V(\omega) \\ \mathfrak{g}\text{-inv.}}} W$ is still proper submod.