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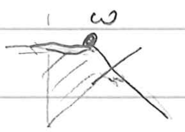
Summary
- Highest weight theory (cont'd)
  - Verma module
- Categorical structure

- Highest weight theory
  (Highest wght vecs, Verma modules)
  (π^ω, V^ω) simple quot. of Verma mod.
  Prop. $\omega \in P_{++} \Rightarrow \dim V^\omega < \infty$
  Step 1 $V^\omega = \bigcup_{\substack{\mathfrak{s}_\alpha\text{-inv} \\ \text{fin dim}}} W$ $\quad (\Rightarrow \pi^\omega_{E_\alpha}, \pi^\omega_{F_\alpha}$ loc. nilpot.)
  RHS is $\mathfrak{g}$-inv
  $\because$ $\mathfrak{g} \otimes W \to V^\omega$ is $\mathfrak{s}_\alpha$-equivar.
  $\omega \in P_{++} \Rightarrow W = \langle F_\alpha^k v : k = 0, \cdots, \omega(H_\alpha) \rangle$
  is $\mathfrak{s}_\alpha$-inv.
  Step 2 $\mathrm{Supp}(V^\omega) = \{\lambda : V^\omega_\lambda \neq 0\}$ is W-inv.
  $\because$ s_α-inv. by Step 1
  Step 3. $\mathrm{Supp}(V^\omega)$ is finite.
  $\because$ $\mathrm{Supp}(V^\omega) \cap P_{++} \subset \{\omega - \sum_{i=1}^n k_i \alpha_i : k_i \geq 0\}$
  E.g.  A_2 is $\begin{cases} \text{finite} \\ \text{fund. dom for } W. \end{cases}$

- Categorical structure
  Overall motivation
  G (cpt) grp $\rightsquigarrow$ "monoidal category" $\mathcal{C}$ & "fiber functor" F.
  s.t. $\begin{cases} (\mathcal{C}, F) \text{ can recover } G \\ \text{abstract characterization of } (\mathcal{C}, F) \end{cases}$

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Ex. $G \rightsquigarrow \mathcal{C}$: category of finite G-sets. (G-Sets$_f$)
$(X, \alpha : G \curvearrowright X)$, $(X \times Y, \alpha \times \beta)$
$F(X, \alpha) = X$.

Ex. $G \rightsquigarrow \mathcal{C}$: category of fin. dim unitary reps $(H, \pi : G \to U(H))$., $(H \otimes H', \pi \otimes \pi')$ (Rep G)
$F(H, \pi) = H$.

How does "abstract characterization" work?

Ex. Galois theory
K field, $\bar{K}$ alg. closure of K.
$\mathcal{C}$ = category of finite étale algs $/K$.
(A étale $\Leftrightarrow$ $A \simeq L_1 \times \cdots \times L_m$, L_i fin. sep$/K$
$F(A) = \mathrm{Hom}_{K\text{-alg}}(A, \bar{K})$. , $F(A \otimes B) = F(A) \times F(B)$
$F(A)$ has left action of $\mathrm{Gal}(\bar{K}/K) = \mathrm{Aut}_{K\text{-alg}}(\bar{K})$
in fact $\mathrm{Gal}(\bar{K}/K) =$ "$\mathrm{Aut}(F)$", ($\mathcal{C} \simeq$ Gal-Sets

Ex. 3-dim QFT $A \doteq (A(O))_{O \subset \mathbb{R}^3}$ fam. of algs.
$\rightsquigarrow \mathcal{C}$: "rep. of A"
(minor cond. on A) $\rightsquigarrow$ $\mathcal{C} \simeq \mathrm{Rep}\, G$ for some G.
Doplicher-Roberts

Ex. étale fund. grp. of scheme (Grothendieck)
X : (conn, loc. Noetherian) $x : \mathrm{Spec}(\bar{K}) \to X$
$\mathcal{C}$: cat. of finite étale schemes $/X$
$F(Y)$ = "fiber of $Y \to X$ over x"
$= \left\{ \begin{array}{ccc} \mathrm{Spec}(\bar{K}) & \to & Y \\ & \searrow^{x} & \downarrow \\ & & X \end{array} \right\}$

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<u>Category.</u> $\mathcal{C}$ is given by

- objects $X, Y \cdots \in \mathrm{Ob}(\mathcal{C})$
- morphisms $S, T \cdots \in \mathrm{Mor}(X, Y)$ $\quad (X, Y \in \mathrm{Ob}(\mathcal{C}))$

$\mathrm{Id}_X \in \mathrm{End}(X) = \mathrm{Mor}(X, X)$

$(S_1 S_2) S_3 = S_1 (S_2 S_3)$, $\quad \mathrm{Id}_Y S = S = S\, \mathrm{Id}_X$.

Ex. $\mathcal{C} = (G\text{-Sets})$

$\mathrm{Mor}((X, \alpha), (Y, \beta)) = \{ f : X \to Y,\ \beta_g f = f \alpha_g \}$

Ex. $\mathcal{C} = \mathrm{Rep}\, G$

$\mathrm{Mor}((H^1, \pi^1), (H^2, \pi^2)) = \{ T : H^1 \to H^2$ lin.

$\pi^2_g T = T \pi^1_g \}$

<u>Functor</u> $F : \mathcal{C} \to \mathcal{C}'$ is given by

- maps $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$

$\mathrm{Mor}(X, Y) \to \mathrm{Mor}(F(X), F(Y))$

s.t. $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$, $\quad F(ST) = F(S) F(T)$

<u>Natural transform</u> $\phi : F \to F'$ is given by

morphisms $\phi_X : F(X) \to F'(X)$ s.t.

$F, F' : \mathcal{C} \to \mathcal{C}'$

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"] \arrow[d, "F(T)"'] \arrow[dr, phantom, "\circlearrowleft"] & F'(X) \arrow[d, "F'(T)"] \\ F(Y) \arrow[r, "\phi_Y"'] & F'(Y) \end{tikzcd} \qquad \forall\, T \in \mathrm{Mor}(X, Y)$$

Nat. iso. : all ϕ_X are invertible.

Ex.

$\mathcal{C} = (G\text{-Sets}_f)$, $\quad \mathcal{C}' = (\mathrm{Sets}_f)$ $\quad F(X, \alpha) = X$.

$g \in G \rightsquigarrow \phi^g \in \mathrm{Aut}(F)$ by $\phi^g_{(X,\alpha)} = \alpha_g$ as

map $F(X, \alpha) \to F(X, \alpha)$

Ex. $\mathcal{C} = \mathrm{Vec}_K = \mathcal{C}'$ $\quad F(V) = V$, $\quad F'(V) = V^{**}$

$\phi : F \to F'$ by $\phi_V(x)(\xi) = \xi(x)$ for $\xi \in V^*$