Summary

- Monoidal cat
- Tannaka-Krein duality

<u>Mon</u>. <u>cat</u>. $\mathcal{C}$ is given by

- cat $\mathcal{C}$
- functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $\quad (X, Y) \mapsto X \otimes Y$.
- dist. ob $1 \in \mathrm{Ob}(\mathcal{C})$ "mon unit"
- nat. isoms $1 \otimes X \overset{\lambda}{\simeq} X \overset{\rho}{\simeq} X \otimes 1$

$$\Phi : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

s.t.

$$\begin{tikzcd}[column sep=small]
((X \otimes Y) \otimes Z) \otimes W \arrow[rr] \arrow[d] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[d] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[dr] & \circlearrowleft & X \otimes ((Y \otimes Z) \otimes W) \arrow[dl] \\
& X \otimes (Y \otimes (Z \otimes W)) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
(X \otimes 1) \otimes Y \arrow[rr] \arrow[dr] & & X \otimes (1 \otimes Y) \arrow[dl] \\
& X \otimes Y &
\end{tikzcd}$$

etc. comm.

$\mathcal{C}, \mathcal{C}'$ mon cat. - <u>Mon</u> <u>functor</u> $F : \mathcal{C} \to \mathcal{C}'$ is given by

- f'tr F $\mathcal{C} \to \mathcal{C}'$
- nat. iso. $F_2 : F(X) \otimes F(Y) \to F(X \otimes Y)$
- iso. $F_0 : 1_{\mathcal{C}'} \to F(1_{\mathcal{C}})$

with compat. for Φ & Φ', $\lambda, \lambda', \cdots$

Ex. (G-Sets) $\quad (X \times Y, (\alpha_g \times \beta_g)_g)$

RepG $\quad (H \otimes H', (\pi_g \otimes \pi'_g)_{g \in G})$

Fiber f'trs to Sets / Hilb_f are mon. f'tr.

<u>Nat</u>. <u>trans</u> of mon. f'trs $F, F' : \mathcal{C} \to \mathcal{C}'$

nat trans $\phi_X : F(X) \to F'(X)$ compat with F_2 & F'_2.

Ex. $F : \mathrm{Rep}\,G \to \mathrm{Hilb}_f$

$g \in G \rightsquigarrow \phi^g : F \to F$ by $\phi^g_{(H,\pi)} = \pi_g : H \to H$

this is aut. as mon. ftr i.e.

$\phi^g_{(H\otimes H', \pi\otimes\pi')} = \pi_g \otimes \pi'_g = \phi^g_{(H,\pi)} \otimes \phi^g_{(H',\pi')}$

T-K duality.

Pt 1. ((Rep G, F) recovers G) $F : \mathrm{Rep}\,G \to \mathrm{Hilb}_f$

satisfies $G \simeq \mathrm{Aut}^{\otimes}(F) \leftarrow$ nat. unitary aut. of F.

Pt 2 (possibility of (Rep G, F))

$\mathcal{C}$: $\mathbb{C}$-lin. mon. cat with • duality, • invol. on mor $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(Y,X)$ $T \mapsto T^*$.

• symmetric br. $c : X\otimes Y \xrightarrow{\sim} Y \otimes X$

$F : \mathcal{C} \to \mathrm{Hilb}$ compat w/ duality, invol, br

$\Rightarrow \exists\, G$ s.t. $\mathcal{C} \overset{\otimes}{\simeq} \mathrm{Rep}\,G$, $F \leftrightarrow$ fiber ftr of G

Pt 1.

Step 1 $\mathrm{Aut}^{\otimes}(F)$ is cpt grp.

$\because$ $\mathrm{Aut}^{\otimes}(F) < \mathrm{Aut}(F) \simeq \prod_{(H_i,\pi_i) : \mathrm{Irrep}\,G} U(H_i)$

any nat iso $\phi : F \to F$ is det'ed by

$\phi_{(H_i,\pi_i)}\ H_i \to H_i$

Step 2. $G \to \mathrm{Aut}^{\otimes}(F)$ is cont inj.

(so $G < \mathrm{Aut}^{\otimes}(F)$)

S3.? Step 5 $\mathrm{Aut}^{\otimes}(F)/G = \{*\}$.

Facts • Stone-Weierstrass th'm.

X cpt, $\mathcal{A} \subset C(X)$ subalg $f \in \mathcal{A} \to \bar{f} \in \mathcal{A}$

sep. pts of X $\Rightarrow \forall f \in C(X), \varepsilon > 0 \cdots$

• Haar measure on G

S3. $\phi \in \mathrm{Aut}^{\otimes}(F) \rightsquigarrow \phi(\mathbb{C}, 1) = 1$, $\phi(\overline{H}, \overline{\pi}) = \overline{\phi(H,\pi)}$

$\phi(\mathbb{C},1) \otimes \phi(\mathbb{C},1)$ "=" $\phi(\mathbb{C},1)$

$(\phi(\overline{H},\overline{\pi}) \otimes \phi(H,\pi))R = R\,\phi(\mathbb{C},1) = R.$

for $R : \mathbb{C} \to \overline{H} \otimes H$, $\lambda \mapsto \lambda \sum_{i=1}^{d} \overline{e_i} \otimes e_i$

$(e_i)_{i=1}^{d}$ ONB.

$(X_{ij})_{i,j=1}^{d} \leftrightarrow \phi(H,\pi)$, $(X'_{\overline{i},\overline{j}}) \leftrightarrow \phi(\overline{H}, \overline{\pi})$ (unitarity)

$\Rightarrow \sum_j X'_{\overline{i}\overline{j}} X_{kj} = \delta_{ik} \Rightarrow X'_{\overline{i}\overline{j}} = \overline{X_{ij}}$.

S4. $\mathcal{O}(G)$: alg. of mat. coeffs.

$\mathcal{O}(G) \subset C(\mathrm{Aut}^{\otimes}(F))$ dense

$f^{\pi}_{\xi,\eta}(\phi) = (\phi(H,\pi)\eta, \xi)$, $\xi, \eta \in H$

- subalg : $f^{\pi}_{\xi,\eta} f^{\pi'}_{\xi',\eta'} = f^{\pi\otimes\pi'}_{\xi\otimes\xi',\eta\otimes\eta'}$
- closed under $f \rightsquigarrow \overline{f}$: $\overline{f^{\pi}_{\xi,\eta}} = f^{\overline{\pi}}_{\overline{\xi},\overline{\eta}}$

$\overline{(\phi(H,\pi)\eta, \xi)} = (\xi, \phi(H,\pi)\eta)$
$= (\overline{\phi(H,\pi)\eta}, \overline{\xi})$

$\overline{\phi(H,\pi)\eta} = \overline{\phi(H,\pi)}\,\overline{\eta} = \phi(\overline{H},\overline{\pi})\overline{\eta}$

- Sep. pts of $\mathrm{Aut}^{\otimes}(F)$. $\phi \neq \phi' \Rightarrow$ diff for some ξ, η.

S5. $f \to (\phi \mapsto \int f(\phi \cdot g)\, d\mu(g))$

is contr. $C(\mathrm{Aut}^{\otimes}(F)) \to C(\mathrm{Aut}^{\otimes}(F)/G)$

sends $\mathcal{O}(G)$ to $\mathbb{C}$.

$\Rightarrow C(\mathrm{Aut}^{\otimes}(F)/G)$ is 1-dim.