MAT4270: EXERCISE SET 2

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Problem 1. When X is a complex matrix, let $\bar{X}$ denote the entry-wise complex conjugate of X. Show that

$$\mathfrak{su}_n = \{X \in M_n(\mathbb{C}) \mid \bar{X}^t = -X, \operatorname{Tr} X = 0\}$$

is the Lie algebra of $\mathrm{SU}(n)$.

Hint: you want to relate the two defining conditions $\bar{U}^t U = I_n$ and $\det U = 1$ of $\mathrm{SU}(n)$ to the above conditions for X by the ansatz $U \sim I_n + \epsilon X$.

Problem 2. Give a concrete isomorphism between $\mathfrak{su}_2$ and $\mathfrak{so}_3$.

Problem 3 (Exercise 8.40)**.** The exponential map $\exp\colon \mathfrak{g} \to G$ is:

- surjective for $G = \mathrm{GL}_n(\mathbb{C})$,
- not surjective for $G = \mathrm{GL}_n^+(\mathbb{R}) = \{A \in \mathrm{GL}_n^+(\mathbb{R}) \mid \det A > 0\}$, or $G = \mathrm{SL}_2(\mathbb{C})$.

Problem 4. For each $g \in G$, the adjoint map $\mathrm{Ad}_g\colon h \mapsto ghg^{-1}$ on G preserves e. Hence we get the induced linear transform $\mathfrak{g} \to \mathfrak{g}$ that we again denote by Ad_g. The transforms $(\mathrm{Ad}_g)_{g \in G}$ give a representation of G on $\mathfrak{g}$, hence we get a representation $(\mathrm{ad}_X)_{X \in \mathfrak{g}}$ of $\mathfrak{g}$ on $\mathfrak{g}$. Thus, ad_X is defined by

$$\mathrm{ad}_X(Y) = \frac{d}{dt}\mathrm{Ad}_{\exp(tX)}(Y)\Big|_{t=0}$$

for $Y \in \mathfrak{g}$. Show that we have

$$\mathrm{ad}_X(Y) = [X, Y].$$

(Recall that the bracket $[X, Y]$ was defined through the identification of $\mathfrak{g}$ with the left invariant vector fields on G.)

Problem 5 (Exercise 9.7)**.** If G is connected and nilpotent ($\mathfrak{g}$ is nilpotent), then the exponential map $\exp\colon \mathfrak{g} \to G$ is surjective.

Hint: by hypothesis, the Taylor power series for e^X and the Baker–Campbell–Hausdorff formula become finite sums.

Problem 6 (Section 10.1)**.** Classify the real Lie algebras of dimension 2.

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