1. Finite groups

Notation. G: finite group, X: finite G-set

Problem 1 (double transitivity)**.** Suppose that G acts *transitively* on X. Let $C(X) = \langle \delta_x \mid x \in X \rangle$ be the space of complex functions on X, and consider the induced action $G \curvearrowright C(X)$. Moreover, let V be the invariant complement of the trivial subrepresentation $\mathbb{C}1_X \subset C(X)$. Explain that the the action of G on V is irreducible if and only if $G \curvearrowright X$ is *doubly transitive* in the sense that

$$\forall x \neq x', y \neq y' \exists g \colon gx = y, gx' = y'.$$

(That is, G acts transitively on $X \times X \setminus \{(x,x) \mid x \in X\}$.)

Reference. [Ser77, Section 2.3, Exercise 2.6]

Problem 2 (integrality of characters)**.** Let R be the set of class functions on G which take values in algebraic integers. Explain the *integrality* of R and its application to an estimate of dimension of irreducible representations.

Reference. [Ser77, Section 6.5]

Problem 3 (fixed point counting and positivity)**.** For $g \in G$, let $f(g)$ be the number of fixed points,

$$f(g) = \left|\{x \in X \mid gx = x\}\right|.$$

Show that f is *positive definite*: that is, if $(g_i)_{i \in I}$ is a family of elements of G and $(c_i)_{i \in I}$ is a family of complex numbers on the same index set, we have

$$\sum_{i,j \in I} c_i \bar{c}_j f(g_j^{-1} g_i) \geq 0.$$

Hint: express $f(g)$ as the trace of some matrix, and use $\mathrm{Tr}(A^* A) \geq 0$ if A^* is the conjugate transpose of A.

Also: $f'(g) = |X| - f(g)$ is *conditionally negative definite*: if the coefficients $(c_i)_i$ satisfy $\sum_i c_i = 0$, then

$$\sum_{i,j \in I} c_i \bar{c}_j f'(g_j^{-1} g_i) \leq 0.$$

This implies that $\exp(-f'(g))$ is positive definite.

Reference. [BO08]

2. Lie algebras

Notation. K: field of characteristic 0 (can be $\mathbb{C}$), $\mathfrak{g}$: finite dimensional Lie algebra over K with bracket $[x, y]$

Problem 4 (Levi's theorem)**.** Explain that there is a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathrm{Rad}(\mathfrak{g})$. Also, give an nontrivial example of this decomposition ($\mathfrak{s} \neq 0 \neq \mathrm{Rad}(\mathfrak{g})$).

Reference. [FH91, Section E.1]

Problem 5 (Ado's theorem)**.** Show that there is an integer $n > 0$ and an injective Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}_n(K)$.

Bonus: this is still true for positive characteristic, and the proof is shorter.

Reference. [FH91, Section E.2] (characteristic 0), [Bou07, Section 1.7, Exercise] (positive characteristic)

Problem 6 (Casimir operator of $\mathfrak{sl}_2$)**.** Consider the element

$$C = \frac{1}{8} H^2 + \frac{1}{4} H + \frac{1}{2} FE$$

in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(K))$.

- Explain the reason that if (π, V) is a representation of $\mathfrak{sl}_2(K)$, the endomorphism $\pi(C) \in \mathrm{End}(V)$ is a scalar multiple of the identity map.
- Compute the corresponding scalar for the natural representation of $\mathfrak{sl}_2(K)$ on the space homogeneous polynomials of degree n in two variables, $\langle x^n, x^{n-1} y, \ldots, y^n \rangle$.
- Express C using the standard basis of $\mathfrak{su}_2$.
- Use the identification $\mathfrak{su}_2 \simeq \mathfrak{so}_3$, express C as a differential operator on the unit sphere $S^2 \subset \mathbb{R}^3$.

Problem 7 (geometric realization of representations)**.** Explain geometric realization of the irreducible representations of SL_2 on the complex projective space $\mathbb{P}^1(\mathbb{C})$.

- find a representation of $\mathfrak{sl}_2(\mathbb{C})$ as algebraic vector fields on $\mathbb{P}^1(\mathbb{C})$.
- what are the holomorphic line bundles on $\mathbb{P}^1(\mathbb{C})$ (usually denoted $\mathcal{O}(n)$ for $n \in \mathbb{Z}$), and what are the space of holomorphic sections $H^0(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n)) = \Gamma(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$?
- describe the first cohomology $H^1(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$, either through holomorphic differential forms in these coefficients, or through the Čech cohomology for the decomposition $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup_{\mathbb{C}^\times} \mathbb{C}$.
- identify these with the spaces of homogeneous polynomials in two variables, and compare the actions of $\mathfrak{sl}_2$ (or SL_2).
- interpret this construction as the Borel–Weil–Bott construction.

Reference. [Sep07]

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