2021 - 08 - 24 no. 1

Motivation behind representation theory

"understand symmetry through linear algebra"

Symmetry : modelled by group

X : math. obj

⇝ $G = \{ \text{automorphisms } X \xrightarrow{\sim} X \}$.

(can compose & has inverse)

still abstract algebraic system.

linear algebra : vectors & matrices.

⇝ we can compute various (numerical) invariants

e.g. eigenvalues, trace, determinant, ...

Guiding principle : "linearize" things.

Nonlinear structure ⟶ linear structure.

1. elements of a group ⇝ representation by
$g \in G$ matrices $(X^g_{i,j})_{i,j}$

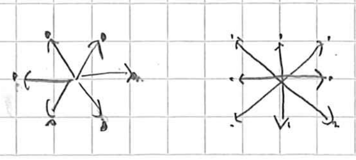
2. continuous group ⇝ infinitesimal model
(Lie group) (Lie algebra)

e.g. vector fields e^{tX} ⇝ derivations X

Benefit : get access to eigenvalues of matrices.

1. understand matrix representations π through traces (<u>character functions</u> $G \xrightarrow{\chi_\pi} \mathbb{C}$)

2. extract combinatorial structures from Lie algebras (<u>root systems</u>)



Overview of the course

Part 1 Linear representation of finite groups

Main ref. Serre (same title)

- basic concepts on complex representation.
  irreducible decomposition, tensor product, ...
- character theory
  "# of representations = # of conjugacy classes."

Part 2 Lie algebras

Main ref. Fulton-Harris, Rep. theory: a first course

- Lie groups $\leftrightarrow$ Lie algebras
- general structure of Lie algebras
  solvable vs. semisimple $\nwarrow$ rigid.
  $\uparrow$ close to "commutative"; soft

Part 3 classification of complex simple Lie algebras

Main ref. Fulton-Harris

- fundamental examples $\mathfrak{sl}_2$, $\mathfrak{sl}_3$
- root system.

Part 1. Lin. rep. of fin. grps

Framework: <u>finite</u> <u>dimensional</u> <u>complex</u> vector spaces V, W, $\mathbb{C}^d$, ...

why? : we can talk about eigendecompositions & eigenvalues for lin. transforms. $T: V \to V$

Notation

$$\mathrm{End}(V) = \{ T : V \to V \text{ linear} \} \simeq M_d(\mathbb{C})$$

$$GL(V) = \{ T \in \mathrm{End}(V), \text{ invertible} \} \simeq GL_d(\mathbb{C})$$

$$d = \dim V$$

G : finite group

Def. a linear representation of G is given by

- (fin. dim.) vector space V "underlying sp."
- group homomorphism $\pi : G \to GL(V)$

i.e. collection of transforms. $\pi_g : V \to V$

invertible & $\pi_g \pi_h = \pi_{gh}$ $g, h \in G$

matrix form : d-dim rep. is given by

$(X^g_{ij})_{i,j=1}^{d}$ for $g \in G$, $\sum_k X^g_{ik} X^h_{kj} = X^{gh}_{ij}$

Not. $\pi_g v$, gv for $\pi_g(v)$ $v \in V$

so $\pi_g \pi_h = \pi_{gh}$ becomes $g(hv) = (gh)v$.

Examples

1. permutation representation

$G = S_n = \{ \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \text{ bij} \}$

n-th symmetric group.

$V = \langle e_1, \ldots, e_n \rangle \simeq \mathbb{C}^n$

$\pi_\sigma e_i = e_{\sigma i}$ $\pi_\sigma \leftrightarrow$ permutation matrix for σ

2. $G = \mathbb{Z}/n\mathbb{Z} = \{ [j] : j \in \mathbb{Z}, [j] = [k] \text{ iff } n \mid j-k \}$

cyclic group

fix $m \in \{0, 1, \ldots, n-1\}$

$\varphi^{(m)}_{[j]} = e^{\frac{2\pi i m j}{n}} \in \mathbb{C} = M_1(\mathbb{C})$

defines a 1-dim rep. $e^{(m)}_{[j]} e^{(m)}_{[k]} = e^{(m)}_{[j+k]}$.

Prop. $\varphi^{(0)}, \ldots, \varphi^{(n-1)}$ exhaust the 1-dim reps. of $G = \mathbb{Z}/n\mathbb{Z}$

Proof. • 1-dim rep. $G \xrightarrow{\psi} \mathbb{C}^\times = GL_1(\mathbb{C})$ is determined by $\zeta = \psi[1]$, $\zeta^n = 1$.

$$\psi[j] = \psi(\underbrace{[1]\cdot[1]\cdot\cdots\cdot[1]}_{j \times}) = \psi[1]\,\psi[1]\cdots\psi[1] = \zeta^j$$

• $\varphi^{(0)}_{[1]}, \ldots, \varphi^{(n-1)}_{[1]}$ exhaust such ζ

Questions we want to address:

- how do we compare representations.
- what are the fundamental building blocks? (and how do we produce other reps.?)
- classification of reps by <u>characters</u>

$$\chi_\pi(g) = \mathrm{Tr}(\pi_g).$$