Notn. $G$ : finite group

$(\pi, V)$, $(\pi', V')$, $\cdots$ : reps. of $G$

comparison of reps.

Def. an <u>intertwiner</u> (or <u>G-homomorphism</u>) from $(\pi, V)$ to $(\pi', V')$ :

a linear map $T : V \to V'$ s.t.

$$T \pi_g v = \pi'_g T v \quad (v \in V, g \in G)$$

- $(\pi, V)$ and $(\pi', V')$ are <u>isomorphic</u> when there is an invertible intertwiner
  write $(\pi, V) \simeq (\pi', V')$, $\pi \simeq \pi'$, $V \simeq V'$, ..
- $(\pi, V)$ is a <u>subrepresentation</u> of $(\pi', V')$ when $\exists$ injective intertwiner $T : V \to V'$

Rem. up to isom. a subrep of $(\pi', V')$ is the same thing as a <u>G-invariant subspace</u> of $V'$ : i.e. a subsp. $W < V'$ s.t. $g \in G$, $w \in W \Rightarrow gw \in W$.

Example $G = \mathbb{Z}/n\mathbb{Z}$ as before

$V' = \langle e_1, \cdots, e_n \rangle$, $\pi'_{[j]} e_k = e_{k+j \ (\mathrm{mod}\ n)}$.

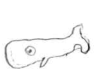
i.e. restrict perm. rep of $S_n$ to subgrp. $G \simeq \langle (1\ 2\ \cdots\ n) \rangle$

fix $m \in \{0, \cdots, n-1\}$

$$W_m = \langle \sum_{k=1}^{n} \exp\left(\frac{2\pi i m k}{n}\right) e_k = w_m \rangle < V'$$

then we have $\pi'_{[j]} w_m = \exp\left(\frac{2\pi i m j}{n}\right) w_m$

So $W_m$ is G-inv., $(\pi', W_m) \simeq (\varphi^{(m)}, \mathbb{C})$



how to combine representations

Def. $(\pi, V)$, $(\pi', V')$ reps of $G$.

$\rightsquigarrow$ their <u>direct</u> <u>sum</u> is given by

- underlying vec. sp. $V \oplus V'$
- $\pi \oplus \pi' : G \to GL(V \oplus V')$, $g \mapsto \pi_g \oplus \pi'_g$.

so $g(v \oplus v') = gv \oplus gv'$.

in matrix form $(X^g_{i,j})_{i,j=1}^m$, $(Y^g_{k,l})_{k,l=1}^n$.

$$\rightsquigarrow \begin{bmatrix} X^g & 0 \\ 0 & Y^g \end{bmatrix} \in M_{m+n}(\mathbb{C})$$

Ex. in the prev. example, we have

$(\pi', V') \simeq (\pi', W_0) \oplus \cdots \oplus (\pi', W_{n-1})$

from $V' = W_0 \oplus \cdots \oplus W_{n-1}$

i.e. any $v' = \sum_{j=1}^{n} \alpha_j e_j$ has unique

expression $v' = \sum_{m=0}^{n-1} \beta_m u_m$

(try this for $n = 2, 3$ ; this is related to discrete Fourier analysis)

decomposition of representations

suppose $W$ is an invar. subsp. of $(\pi, V)$

$\rightsquigarrow$ can we find another invar. subsp. $W'$

s.t. $V = W \oplus W'$ ? i.e. $(\pi, V) \simeq (\pi, W) \oplus (\pi, W')$?

warning : for general rings & modules

this is not guaranteed

∵ but for lin reps of fin. grps this is possible :

Th'm $(\pi, V)$ rep of $G$, $W \subset V$ invar. subsp.

then $\exists\ W' \subset V$ invar, $V = W \oplus W'$

(any invar subsp. has an invar. complement ; we can split off subrepresentations as direct summands )

recall : <u>Hermitian</u> <u>inner</u> <u>product</u> :

$(v, v') \in \mathbb{C}$ for $v, v' \in V$ s.t.

- linear in $v$, conjugate lin. in $v'$
- $(v', v) = \overline{(v, v')}$
- $(v, v) \geq 0$ for $v \in V$, $(v, v) = 0 \Leftrightarrow v = 0$

key part of the proof of Th'm :

$V$ has an invariant Hermitian inn. prod :

$(gv, gv') = (v, v')$ $(v, v' \in V,\ g \in G)$

∵ take any inn. prod. $(v, v')_0$

put $(v, v') = \frac{1}{|G|} \sum_{h \in G} (hv, hv')_0$

i.e. average $(\sim, \sim)_0$ by the action of $G$.

then $(\sim, \sim)$ is invariant :

$(gv, gv') = \frac{1}{|G|} \sum_h (hgv, hgv')_0$

$\underset{h' = hg}{=} \frac{1}{|G|} \sum_{h' \in G} (h'v, h'v')_0 = (v, v')$

(cont.) rest of the proof of Th'm.

take $W' = \{ m' \in V : \forall m \in W \quad (m, m') = 0 \}$

orthogonal complement of $W$.

- $V = W \oplus W'$ (standard lin. alg.)

- $W'$ is invariant under $G$

we need to check $g m' \in W'$ for $(m' \in W')$

i.e. $(m, g m') = 0$ for all $m \in W$.

by the invariance of $(\sim, \sim)$

$$(m, g m') = (g^{-1} m, \underbrace{g^{-1} g}_{e} m') = (g^{-1} m, m')$$

$$g^{-1} m \in W \Rightarrow (g^{-1} m, m') = 0$$ $\boxtimes$

Def. $(\pi, V)$ is <u>simple</u> (or <u>irreducible</u>) if

$0$ and $V$ are the only invariant subspaces

Cor. of Th'm. any rep $(\pi, V)$ is isom. to

a direct sum of irred. reps

$$(\pi, V) \simeq (\pi_1, V_1) \oplus \cdots \oplus (\pi_m, V_m)$$

$(\pi_i, V_i)$ irred.

$\because$ take $0 \subsetneq W \subsetneq V$ invariant (if any)

$\underset{\text{Th'm}}{\Rightarrow}$ $V = W \oplus W'$ with invar. $W'$

by induction on dimension

$W$ and $W'$ are dir. sums of irred. reps

$\Rightarrow$ so is $V$. $\boxtimes$