We now know how to compare reps (π, V) & (π', V')

(by a lin. map $T : V \to V'$ s.t. $\pi'_g T v = T \pi_g v$)

can we distinguish representations ?

Ex. YES for 1-dim reps. ($\equiv G \xrightarrow{\pi, \pi'} \mathbb{C}^\times$)

π_g and T will commute, so $T \neq 0 \Rightarrow \pi_g = \pi'_g$

Th'm (Schur's lemma)

(π, V), (π', V') <u>irreducible</u> reps of G

$T : V \to V'$ intertwiner

1) if $\pi \not\simeq \pi'$, then $T = 0$

2) if ($\pi \simeq \pi'$ and) $T \neq 0$, any other intertwiner $S : V \to V'$ is a scalar multiple of T

Proof overview

- T intertw. $\Rightarrow \operatorname{Im} T \subset V'$, $\operatorname{Ker} T \subset V$ are G-invar.
- π, π' irred. $\Rightarrow \operatorname{Im} T$, $\operatorname{Ker} T$ have to be $0, V, V'$

$\Rightarrow T = 0$ or bijective ($\pi \simeq \pi'$)

Step 1 G-invariance of $\operatorname{Im} T$

want : $\forall v \in V, g \in G \; \exists w \quad \pi'_g T v = T w$

we can take $w = \pi_g v$.

Step 2 same for $\operatorname{Ker} T$

want : $\forall v \in \operatorname{Ker} T, g \in G \quad T \pi_g v = 0$

use $\pi'_g T v = T \pi_g v$

Step 3 $T \neq 0 \Rightarrow T$ injective

$\operatorname{Ker} T \neq V$ by assumption

by the irreducibility of (π, V), $\operatorname{Ker} T = 0$

Step 4 $\quad T \neq 0 \Rightarrow T$ surjective

otherwise $\operatorname{Im} T = 0$ by the irreducibility of (π', V') $\Rightarrow T = 0$

Step 3 & Step 4 gives claim 1)

Step 5 $\quad T$ bijective, $S : V \to V'$ intertwiner

$\exists\, \alpha \in \mathbb{C} \quad S = \alpha T$

$\because$ Put $A = S T^{-1} : V' \to V'$

take any eigenvalue $\alpha \in \mathbb{C}$ of A.

$\rightsquigarrow A - \alpha \operatorname{Id}_{V'}$ is an intertw.

$\because \quad \pi'_g S T^{-1} v' = S \pi_g T^{-1} v' \underset{\uparrow}{=} S T^{-1} \pi'_g$ for A.

$T^{-1}(T\pi_g = \pi'_g T)T^{-1}$

$A - \alpha \operatorname{Id}_{V'}$ has nontriv. ker $\underset{\text{Step 3}}{\Rightarrow}$ $A - \alpha \operatorname{Id}_{V'} = 0$

i.e. $A = \alpha \operatorname{Id}_{V'} \Leftrightarrow S = \alpha T$ $\quad\square$

Averaging to create intertwiners

intertwiner cond. $\pi'_g T = T \pi_g$ is equiv. to

$\pi'_g T \pi_g^{-1} = T$

observation. • $\operatorname{Ad}_g T = \pi'_g T \pi_g^{-1}$ for $T \in \operatorname{Hom}(V, V')$

$(= \{\text{linear maps } V \to V'\})$ satisfies

$\operatorname{Ad}_g(\operatorname{Ad}_h T) = \operatorname{Ad}_{gh} T$ i.e. $(\operatorname{Ad}_g)_{g \in G}$ is a

representation of G on $\operatorname{Hom}(V, V')$

• T intertw. $\Leftrightarrow$ T is invariant under Ad.

Prop $\quad T \in \operatorname{Hom}(V, V')$ then $\tilde{T} = \frac{1}{|G|} \sum_{g \in G} \operatorname{Ad}_g T$

is an intertw.

Proof $\quad \operatorname{Ad}_g \tilde{T} = \frac{1}{|G|} \sum_{h \in G} \operatorname{Ad}_g(\operatorname{Ad}_h T) = \frac{1}{|G|} \sum_h \operatorname{Ad}_{gh} T$

$\underset{h' = gh}{=} \frac{1}{|G|} \sum_{h'} \operatorname{Ad}_{h'} T = \tilde{T}$

Cor (π, V), (π', V') irred. reps of G

$T \in \mathrm{Hom}(V, V')$ $\tilde{T}$ as above.

1) $\pi \not\cong \pi' \Rightarrow \tilde{T} = 0$

2) $V = V'$, $\pi = \pi' \Rightarrow \tilde{T} = \frac{\mathrm{Tr}\, T}{\dim V} \mathrm{Id}_V$

Proof. 1) from Prop. & Schur's lem. (1).

2) $\tilde{T} = \alpha\, \mathrm{Id}_V$ by Schur's lem (2)

$\alpha \cdot \dim V = \mathrm{Tr}(\alpha\, \mathrm{Id}_V) = \mathrm{Tr}\, \tilde{T} \underset{\uparrow}{=} \mathrm{Tr}\, T$

$\tilde{T} = \frac{1}{|G|} \sum \pi_g T \pi_g^{-1}$, $\mathrm{Tr}(\pi_g T \pi_g^{-1}) = \mathrm{Tr}\, T$

Def. the <u>character</u> of (π, V) is the function

$\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$

i.e. if $(x^g_{ij})_{i,j=1}^{d}$ is the matrix form,

$\chi_\pi(g) = \sum_i x^g_{ii}$

Rem 1) $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$; χ_π is a <u>class function</u> (const. on conjugacy classes)

$\because$ $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}\, B$

2) $\chi_\pi(g) \in \mathbb{Z}[\exp(\frac{2\pi i}{|G|})]$

Examples 1) 1-dim rep $G \xrightarrow{\psi} \mathbb{C}^\times$ $\chi_\psi(g) = \psi(g)$

2) permutation rep. $S_n \xrightarrow{\pi} GL(V)$; $V = \langle e_1, \dots, e_n \rangle$

$\chi_\pi(\sigma) = \#\{ i : \sigma(i) = i \}$

$\hookrightarrow$ Tr (perm. mat for σ) = # of 1's in the diag

Prop. (π, V), (π', V') reps of G

$\chi_{\pi \oplus \pi'} = \chi_\pi + \chi_{\pi'}$

$\because$ $\mathrm{Tr}(\pi_g \oplus \pi'_g) = \mathrm{Tr}(\underset{x^g}{\pi_g}) + \mathrm{Tr}(\underset{Y^g}{\pi'_g})$

$\begin{bmatrix} X^g & 0 \\ 0 & Y^g \end{bmatrix}$

Ex. $G = \mathbb{Z}/n\mathbb{Z}$ $\quad \pi$ perm rep. on $V = \langle e_1, \dots, e_n \rangle$

1-dim reps $\varphi^{(m)}_{[j]} = \exp\left(\frac{2\pi i m j}{n}\right)$

$$\chi_\pi([j]) = \begin{cases} n & [j] = [0] \\ 0 & [j] \neq [0] \end{cases}$$ by counting fixed points

$$\sum_{m=1}^{n} \varphi^{(m)}_{[j]} = \begin{cases} n = 1 + \dots + 1 & j = 0 \\ 0 & [j] \neq [0] \end{cases}$$

reflecting $\pi \simeq \varphi^{(1)} \oplus \dots \oplus \varphi^{(n)}$

Orthogonality of irreducible characters

Not'n $\varphi, \psi : G \to \mathbb{C}$ functions

$$(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

Hermitian inner product for the uniform prob. meas.

Th'm. 1) (π, V) irred. rep. of $G \Rightarrow (\chi_\pi, \chi_\pi)_{L^2 G} = 1$

2) (π', V') another irred. rep. of G, $\pi \not\simeq \pi'$

$\Rightarrow (\chi_\pi, \chi_{\pi'}) = 0$

Proof tomorrow

What's great? : $(\chi_\pi, \chi_\pi) = 1 \Leftrightarrow \pi$ irred.

ex. if $\pi \simeq \pi_1 \oplus \pi_2$, $\pi_1 \not\simeq \pi_2$ irred.

$\Rightarrow (\chi_\pi, \chi_\pi) = \sum_{i,j} (\chi_{\pi_i}, \chi_{\pi_j}) = \sum_i (\chi_{\pi_i}, \chi_{\pi_i}) = 2$

Example $G = \mathbb{Z}/n\mathbb{Z}$

irred. characters $\varphi^{(1)}, \dots, \varphi^{(m)}$ as above

$$(\varphi^{(k)}, \varphi^{(l)})_{L^2(G)} = \delta_{k,l}$$

$\because \hookrightarrow \frac{1}{n} \sum_{j=0}^{n-1} \varphi^{(k)}_{[j]} \overline{\varphi^{(l)}_{[j]}} = \frac{1}{n} \sum_j \exp\left(\frac{2\pi i (k-l) j}{n}\right)$

$$= \begin{cases} 1 \\ 0 \end{cases}$$