Recall : character $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$

inner prod. $(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$

$\mathrm{Ad}\, T = \pi'_g T \pi_g^{-1}$ for $T \in \mathrm{Hom}(V, V')$

We wanted to prove (orthogonality)

$(\chi_\pi, \chi_\pi)_{L^2(G)} = 1$, $(\chi_\pi, \chi_{\pi'}) = 0$ $(\pi \not\cong \pi')$

for irred. reps π, π'

Rem. if two reps. (π, V) and (π', V') are isomorphic then $\chi_\pi = \chi_{\pi'}$

$\because$ again use $\mathrm{Tr}\, ABA^{-1} = \mathrm{Tr}\, B$

$T : V \to V'$ inv. intertw. $\Rightarrow T \pi_g T^{-1} = \pi'_g$

we'll see $\chi_\pi = \chi_{\pi'} \Rightarrow \pi \cong \pi'$.

Prop. For $\mathrm{Ad}_g(T) = \underline{\mathrm{Ad}_g^{\pi,\pi'}}(T) = \pi'_g T \pi_g^{-1}$

$\chi_{\mathrm{Ad}}(g) = \chi_{\pi'}(g) \overline{\chi_\pi(g)}$

Proof overview

- $\mathrm{Hom}(T, T')$ has a basis of "matrix units"

  $\rightsquigarrow \mathrm{Ad}_g$ is represented by Kronecker product

- π_g^{-1} has the trace $\overline{\chi_\pi(g)}$

Step 0 : take basis $(e_i)_{i=1}^m$ of V

$(X^g_{ij})_{i,j=1}^m$ mat. rep. of π_g

$\left(\pi_g e_j = \sum_i X^g_{ij} e_i \right)$

similarly $(f_k)_{k=1}^n$ for V', $(Y^g_{kl})_{k,l}$ for π'_g.

Step 1 : $\mathrm{Hom}(V, V')$ has a basis $(T^i_k)_{i=1, k=1}^{m \quad n}$

s.t. $T^i_k(\sum \alpha_j e_j) = \alpha_i f_k$

$T^i_k \leftrightarrow$ matrix unit $E^{ik} \in M_{n \times m}(\mathbb{C})$

Step 2 : the diagonal components of Z = mat rep. of Ad are $Z^g_{ik,ik} = Y^g_{kk} X^{g^{-1}}_{ii}$

(cont.) i.e. $Ad_g T^i_k = Y^g_{kk} X^{g^{-1}}_{ii} T^i_k$ + linear combination of the other $T^{i'}_{k'}$

$\therefore$ $\pi'_g T^i_k \pi_g^{-1}$ corresp. to $Y^g \left(\begin{bmatrix} 0 & \cdots & \vdots & \\ \vdots & & 1 & \cdots \\ & \cdots & \vdots & 0 \end{bmatrix} \right) X^{g^{-1}}$

$$= \begin{bmatrix} & Y^g_{kk} X^{g^{-1}}_{ii} & \cdots \\ & \vdots & \\ & \vdots & \end{bmatrix}$$

Step 4 $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$

if λ is an eigenval of π_g, $\lambda^{|G|} = 1$
by $\pi_g^{|G|} = \pi_{g^{|G|}} = \pi_e = Id_V$

$\lambda^{-1} = \bar{\lambda}$ would be an eigenval. of $\pi_g^{-1} = \pi_{g^{-1}}$
by $\lambda^N = 1$

Step 3 $\chi_{Ad}(g) = \chi_{\pi'}(g)\, \chi_\pi(g^{-1})$

$Tr(Ad_g) = \sum_{i,k} Z^g_{ik,ik} \underset{\text{Step 2}}{=} \sum_{i,k} Y^g_{kk} X^{g^{-1}}_{ii} = \left(\sum_k Y^g_{kk}\right)\left(\sum_i X^{g^{-1}}_{ii}\right)$

Proof of the orthogonality ; overview

1. $(\chi_{\pi'}, \chi_\pi)_{L^2 G} = \frac{1}{|G|} \sum_g \chi_{\pi'}(g) \overline{\chi_\pi(g)}$ is the trace of the map $Hom(V, V') \to Hom(V, V')$, $T \mapsto \tilde{T}$
$\tilde{T} = \frac{1}{|G|} \sum_g Ad_g T$ from yesterday

2. Cor. to Schur's lem. (yesterday) says what this map is : zero map when $\pi \not\cong \pi'$
projection to the 1-dim subsp. $\mathbb{C} Id_V$ if $\pi' = \pi$.

Step 1. claim 1 above

$(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \frac{1}{|G|} \sum_g \chi_{\pi'}(g) \overline{\chi_\pi(g)} \underset{\text{Prop.}}{=} \frac{1}{|G|} \sum_g Tr(Ad_g)$

$= Tr\left(\frac{1}{|G|} \sum_g Ad_g \right).$

$\left(\frac{1}{|G|} \sum_g Ad_g \right)(T) = \frac{1}{|G|} \sum_g \pi'_g T \pi_g^{-1} = \tilde{T}$ by def.

Step 2 $\pi \not\cong \pi'$ (irr.) $\Rightarrow (\chi_{\pi'}, \chi_\pi)_{L^2(G)} = 0$

$T \mapsto \tilde{T}$ is the zero map on $\mathrm{Hom}(V, V')$
intertw.

Step 3 $(\chi_\pi, \chi_\pi)_{L^2(G)} = 1$

$\tilde{T} = \frac{\mathrm{Tr}(T)}{\dim V} \mathrm{Id}_V$ (see yesterday's note)

we have $\tilde{\tilde{T}} = \tilde{T}$ so $T \mapsto \tilde{T}$ is an

idempotent map $\mathrm{End}(V) \twoheadrightarrow \mathbb{C}\,\mathrm{Id}_V$ ($\subset \mathrm{End}(V)$)

i.e. rank 1 idempotent $\Rightarrow$ has trace 1. $\boxtimes$

Review of some ingredients

Def. the <u>contragredient</u> (or <u>conjugate</u>) representation of (π, V) is the rep. given by

- underlying space $V^* = \mathrm{Hom}(V, \mathbb{C})$
- transforms $\pi^*_g = \mathrm{Ad}_g$ ($g \in G$) on V^*

i.e. $\varphi \in V^*$, $g \in G \rightsquigarrow \pi^*_g \varphi$ is the functional $v \mapsto \varphi(\pi_g^{-1} v)$ on V

other notations (π^c, V^*), $(\bar{\pi}, \bar{V})$

Def. the <u>tensor</u> <u>product</u> of (π, V) and (π', V'):

- underlying space $V \otimes V'$
- transforms $\pi_g \otimes \pi'_g : v \otimes v' \mapsto \pi_g v \otimes \pi'_g v'$ for $g \in G$.

i.e. matrix pres. is the Kronecker prod. $X^g_{ij} Y^g_{kl}$ if $(X^g_{ij})_{i,j}$ reps. π_g, $(Y^g_{kl})_{k,l}$ repr. π'_g

Rem $\chi_{\pi \otimes \pi'}(g) = \chi_\pi(g)\, \chi_{\pi'}(g)$

Ad on $\mathrm{Hom}(V, V') \simeq \pi' \otimes \pi^*$ on $V' \otimes V^*$

up to identification $v' \otimes \varphi \leftrightarrow (v \mapsto \varphi(v) v')$

X

X

Structure of representations

(π, V) : rep of G

$(\pi_1, V_1), \dots, (\pi_n, V_n)$: irred. rep. of G, $\pi_i \not\simeq \pi_j$ $(i \neq j)$

Def. if $\pi \simeq \underbrace{\pi_1 \oplus \cdots \oplus \pi_1}_{m_1 \times} \oplus \underbrace{\pi_2 \oplus \cdots \oplus \pi_2}_{m_2 \times} \oplus \cdots \oplus \underbrace{\pi_n \oplus \cdots \oplus \pi_n}_{m_n}$

we say π contains π_i with multiplicity m_i

Prop. the numbers $(m_i)_{i=1}^n$ are well-defined. i.e.

if $V_1^{\oplus m_1} \oplus \cdots \oplus V_n^{\oplus m_n} \simeq V_1^{\oplus m_1'} \oplus \cdots \oplus V_n^{\oplus m_n'}$

as representations of G, then $m_i = m_i'$

Proof. $\chi_{\pi' \oplus \pi''} = \chi_{\pi'} + \chi_{\pi''}$ gives $\chi_\pi = \sum_{i=1}^n m_i \chi_{\pi_i}$

By orthogonality $(\chi_{\pi_i}, \chi_{\pi_j})_{L^2 G} = \delta_{ij}$

so $m_i = (\chi_\pi, \chi_{\pi_i})_{L^2 G}$ $\boxtimes$

Cor. $(\chi_\pi, \chi_\pi)_{L^2 G} = 1 \iff \pi$ is irreducible.

$\because$ In the above setting $(\chi_\pi, \chi_\pi) = \sum_{i=1}^n m_i^2$

(we can always assume this by compl. reducibility)

$\sum m_i^2 = 1 \iff \exists i \; m_i = 1, \; j \neq i \; m_j = 0$

$\iff \exists i \; \pi \simeq \pi_i$ $\boxtimes$

Examples 1. $G = \mathbb{Z}/n\mathbb{Z}$ $\quad G \xrightarrow{\pi} \underset{\text{perm. rep}}{GL(V)}$, $V = \langle e_1, \dots, e_n \rangle$

we get $\pi \simeq \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$ from

$$(\chi_\pi, \varphi^{(i)})_{L^2 G} = \frac{1}{n} \sum_{j=0}^{n-1} \underbrace{\chi_\pi([j])}_{n \text{ or } 0} \overline{\varphi^{(i)}([j])} = 1.$$

2. $G = S_n$, $\quad G \xrightarrow{\pi} GL(V)$ perm. rep.

$v_0 = \sum_{i=1}^n e_i \Rightarrow \pi_\sigma v_0 = \sum_{i=1}^n e_{\sigma i} = v_0$

so $V_0 = \mathbb{C} v_0$ is G-invar. ($\simeq$ triv. rep.).

$W = \{ \sum \alpha_i e_i : \sum \alpha_i = 0 \}$ is a G-inv. compl.

(try this) $\pi|_W$ is irred. $(\chi_{\pi|_W} = \chi_\pi - \chi_{\text{triv.}})$