What we know so far about characters

- $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$ class function
- (π, V), (π', V') irred, $\pi \not\cong \pi'$

  $\Rightarrow (\chi_\pi, \chi_{\pi'})_{L^2 G} = 0$ orthogonality.
- (π, V) irred. $\Leftrightarrow (\chi_\pi, \chi_\pi)_{L^2 G} = 1$

Example of irr. chars.

$G = S_3 = \langle e, t = (1,2), c = (1,2,3) : t^2 = e, c^3 = e$
$tc = c^2 t \rangle$

two 1-dim reps.

- trivial rep $\pi_{triv}(\sigma) = 1 \quad \forall \sigma$.
- sign rep. (alternating rep.) $\pi_{sig}(\sigma) = (-1)^{|\sigma|}$

so $\chi_{triv}(\sigma) = 1$, $\chi_{sig}(e) = 1$, $\chi_{sig}(t) = -1$, $\chi_{sig}(c) = 1$

the permutation rep (π, V) with $V = \langle e_1, e_2, e_3 \rangle$

contains π_{triv} with multiplicity 1

$v_0 = e_1 + e_2 + e_3$ is inv. under $\pi_\sigma \quad \sigma \in S_3$.

S_3-invar. compl. : $W = \{ \sum_{i=1}^{3} \alpha_i e_i : \sum \alpha_i = 0 \}$

θ = restr. of π to W.

then $\chi_\pi = \chi_{triv} + \chi_\theta$ from $\pi \cong \pi_{triv} \oplus \theta$

but we know $\chi_\pi(\sigma) = \#\{ i : \sigma i = i \}$

$\Rightarrow \chi_\theta(e) = 2$, $\chi_\theta(t) = 0$, $\chi_\theta(c) = -1$

Claim $(\chi_\theta, \chi_\theta)_{L^2 S_3} = 1$ (hence θ irred.)

$|S_3| = 6$, conjug classes $\{e\}, \{t, (2,3), (1,3)\}$
$\{c, c^2 = (3,2,1)\}$

so $(\chi_\theta, \chi_\theta)_{L^2 S_3} = \frac{1}{6}(1 \cdot 2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$

Def <u>regular</u> <u>representation</u> of G is given by

- underlying space $\mathbb{C}[G] \simeq \langle e_g : g \in G \rangle$
- lin. transforms $\lambda_g e_h = e_{gh}$.

Prop 1 $\chi_\lambda(e) = |G|$, $\chi_\lambda(g) = 0$ $g \neq e$

we are counting the fixed points of $G \to G$, $h \mapsto gh$

Cor 1. $(\pi_1, V_1), \cdots, (\pi_n, V_n)$ the irred. reps of G.

$\pi_i \not\simeq \pi_j$ for $i \neq j$.

Then $\lambda \simeq \pi_1^{\oplus \dim V_1} \oplus \pi_2^{\oplus \dim V_2} \oplus \cdots \oplus \pi_m^{\oplus \dim V_m}$.

$\because$ Suppose $\lambda \simeq \pi_1^{\oplus m_1} \oplus \cdots \oplus \pi_n^{\oplus m_n}$

Then $(\chi_\lambda, \chi_{\pi_i})_{L^2 G} = m_i$ by orthogonality

but $(\chi_\lambda, \chi_{\pi_i})_{L^2 G} = \frac{1}{|G|} |G| \overline{\chi_{\pi_i}(e)} = \overline{\chi_{\pi_i}(e)}$

$= \overline{\mathrm{Tr}\, \mathrm{Id}_{V_i}} = \dim V_i$.

Prop 2 G commutative $\Leftrightarrow$ $\forall$ irred. rep. is 1-dim.

Proof $\Rightarrow$: if (π, V) is a rep. of G,

$(\pi_g)_{g \in G}$ pairwise commute (and diagonalizable)

$\Rightarrow$ $\exists$ basis of joint eigenvectors $(v_i)_i$ in V

$\Rightarrow$ each v_i spans a 1-dim irred. rep of G

$\Leftarrow$: $\lambda \simeq \pi_1 \oplus \cdots \oplus \pi_n$ irr. decomp.

$(\pi_1(g) \oplus \cdots \oplus \pi_n(g))_{g \in G}$ pairwise comm.

$\Rightarrow$ $(\lambda_g)_{g \in G}$ pairwise comm $\Rightarrow$ $g \in G$ pairwise comm.

Rem. $\hat{G} = \{$ 1-dim reps of $G \}$ is a comm grp.

by $(\varphi \cdot \psi)(g) = \varphi(g)\psi(g)$ for $\varphi, \psi : G \to \mathbb{C}$ hom.

(for comm. grp G) $\hat{G}$ is the <u>Pontryagin</u> <u>dual</u> group of G.

2021-09-07 no. 3

Next goal: # of irred reps = # of conjug. classes

we already know $\leq$:

$(\chi_{\pi_i})_i$ for irred. reps are linearly indep.

by orthogonality

$\Rightarrow$ # of irreps $\leq$ dim. of class funcs.

= # of conjug. classes.

So we need to know that $(\chi_{\pi_i})_i$ span the space of class funcs.

Notn. (π, V) rep. of G. $f: G \to \mathbb{C}$ function.

$$\pi_f = \sum_{g \in G} f(g)\, \pi_g \in \mathrm{End}(V)$$

Prop 2. $f: G \to \mathbb{C}$ class function, (π, V) rep.

1. π_f is an intertwiner on V
2. if π is irred., $\pi_f = \frac{|G|}{\dim V} (f, \overline{\chi_\pi})_{L^2(G)} \mathrm{Id}_V$.

Proof 1. want: $\pi_g \pi_f \pi_g^{-1} = \pi_f$.

$$\pi_g \pi_f \pi_g^{-1} = \sum_h f(h)\, \pi_{ghg^{-1}} \underset{(h' = ghg^{-1})}{=} \sum_{h'} f(g^{-1}h'g)\, \pi_{h'}$$

$= \pi_f$. by $f(g^{-1}h'g) = f(h')$.

2. Schur's lemma & part 1. implies $\pi_f = \alpha \mathrm{Id}_V$.

$$\alpha = \frac{\mathrm{Tr}\, \pi_f}{\dim V}$$

$$\mathrm{Tr}\, \pi_f = \sum_g f(g)\, \mathrm{Tr}(\pi_g) = |G|\, (f, \overline{\chi_\pi})_{L^2(G)}.$$

Thm. the irreducible characters form a basis of class functions.

Proof. overview) take the irred. reps

$\pi_1, \dots, \pi_n$ ($\pi_i \not\cong \pi_j$ for $i \neq j$)

enough to see: f class func, $\overline{f} \perp \chi_{\pi_i}$ for all i $\Rightarrow f = 0$.

for $(\sim, \sim)_{L^2 G}$

cont. $\bar{f} \perp \chi_{\pi_i} \Rightarrow (\pi_i)_f = 0 \Rightarrow \lambda_f = 0 \Rightarrow f = 0$.

Step 1 $(\bar{f}, \chi_{\pi_i})_{L^2 G} = 0 \Rightarrow (\pi_i)_f = 0$

use $(f, \overline{\chi_{\pi_i}})_{L^2 G} = 0$ and Prop 2.1.

Step 2 $(\pi_i)_f = 0$ for all $i \Rightarrow \lambda_f = 0$

from $\lambda \simeq \pi_1^{\dim V_1} \oplus \cdots \oplus \pi_n^{\dim V_n}$

i.e., $\exists$ invertible intertw. $T : \mathbb{C}[G] \to V_1^{\dim V_1} \oplus \cdots$

$T \lambda_f = ((\pi_1)_f \oplus \cdots \oplus (\pi_n)_f) T$.

so $(\pi_i)_f = 0 \ \forall i \Rightarrow \lambda_f = 0$

Step 3 $\lambda_f = 0 \Rightarrow f = 0$

$\because \sum_g f(g) e_g = \lambda_f \cdot e_e$.

Cor $|G| = \sum_{i=1}^{k} n_i^2$; $k = \#$ conjug. classes.

$n_i = 1, 2, \cdots$

(dim of irred. rep π_i)

$\because \chi_\lambda = \sum_{i=1}^{n} (\dim V_i) \chi_{\pi_i}$ from Cor 1 (of Prop 1),

with $n = \#$ conj. classes by Thm.

Ex. $G = S_3$ $6 = 1^2 + 1^2 + 2^2$.

π_{triv} π_{sig} θ.

Conceptually $\mathbb{C}[G] \simeq \prod_{i=1}^{k} \mathrm{End}(V_k)$

center $\{\lambda_f : f : \text{class func.}\} \longleftrightarrow \prod_{i=1}^{k} \mathbb{C}\,\mathrm{Id}_{V_i}$