Induction from subgroup.

$H < G$ subgroup ; if (π, V) is a rep. of G

we get a rep $(\pi|_H, V)$ of H by restriction of π.

$\rightsquigarrow$ want to make sense of "inverce" op.

called the <u>induction</u>

Def (1st model of induction)

(θ, W) : rep. of H ; the <u>induced representation</u>

$\mathrm{Ind}_H^G(\theta, W)$ (or $\mathrm{Ind}_H^G \theta$) is given by

- underlying space $\mathrm{Ind}_H^G W = \{ f : G \to W$ map s.t. $\forall g \in G, h \in H : f(gh) = \theta_h^{-1} f(g) \}$
- lin. transforms π_g $(g \in G)$ given by

$$(\pi_g f)(g') = f(g^{-1}g')$$

Prop 1. this is well-defined ; i.e.

1. $f \in \mathrm{Ind}_H^G W \Rightarrow \pi_g f \in \mathrm{Ind}_H^G W$
2. $\pi_{g_1} \pi_{g_2} = \pi_{g_1 g_2}$

Proof 1. $(\pi_g f)(g'h) \underset{\text{def. of } \pi_g}{=} f(g^{-1}g'h) \underset{f \in \mathrm{Ind}_H^G W}{=} \theta_h^{-1} f(g^{-1}g')$

$$= \theta_h^{-1} (\pi_g f)(g')$$

2. $(\pi_{g_1}(\pi_{g_2} f))(g') = (\pi_{g_2} f)(g_1^{-1} g') = f(g_2^{-1} g_1^{-1} g')$

$$= f((g_1 g_2)^{-1} g') = (\pi_{g_1 g_2} f)(g')$$

Prop 2 1. $\mathrm{Ind}_H^G W \simeq W^{|G/H|}$, with concrete isom given by $f \mapsto (f(g_1), f(g_2), \cdots, f(g_k))$

for $g_1, \cdots, g_k$: a system of representatives of the right H-cosets $\left(G = \coprod_{i=1}^{k} g_i H \right)$.

2. up to this isom, π_g moves the i-th copy of W to the j-th copy when $g g_i \in g_j H$

Proof 1. $f \in \mathrm{Ind}_H^G W$ is determined by the values $f(g_1), \dots, f(g_k)$: $f(g_i h) = \theta_h^{-1} f(g_i)$

No restrictions on thes vals. $f(g_i h h') = \theta_{hh'}^{-1} f(g_i) = \theta_{h'}^{-1} f(g_i h)$

2. suppose $f \in \mathrm{Ind}_H^G W$ satisfies $f(g_{i'}) = 0$

for $i' \neq i$ (elem. in the i-th copy of W)

claim : $(\pi_g f)(g_{j'}) = 0$ for $j' \neq j$

i.e. $g^{-1} g_{j'} \in g_{i'} H$ for some $i' \neq i$

if $g^{-1} g_{j'} \in g_i H$ ($\exists h$ $g^{-1} g_{j'} = g_i h$)

then $g g_i = g_{j'} h^{-1} \in g_{j'} H$ so $j' = j$. ⊠

Rem (2nd model of induction)

in terms of the $\mathbb{C}[G]$-modules & $\mathbb{C}[H]$-modules

$\mathrm{Ind}_H^G(\theta, W)$ is given by $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$

for the natural inclusion $\mathbb{C}[H] \subset \mathbb{C}[G]$

space of lin. combinations of $a \otimes m$ $a \in \mathbb{C}[G]$, $m \in W$

with extra rule $ab \otimes m = a \otimes bm$ for $b \in \mathbb{C}[H]$

Example the perm rep. of S_n on $V = \langle e_1, \dots, e_n \rangle$

is $\mathrm{Ind}_{S_{n-1}}^{S_n} \pi_{\mathrm{triv}}$; $S_n / S_{n-1} \simeq \{1, \dots, n\}$

$$[\sigma] \longleftrightarrow \sigma n$$

Th'm (Frobenius reciprocity for induction)

(π, V) rep of G, (θ, W) : rep of H

then there's a natural isom of intertwiner spaces

$$\mathrm{Hom}_H((\theta, W), (\pi|_H, V)) \simeq \mathrm{Hom}_G(\mathrm{Ind}_H^G(\theta, W), (\pi, V))$$

Proof. fix $T \in \mathrm{Hom}_H(W, V)$, representatives

$e = g_1, \dots, g_k$ of right H-cosets.

and write $\pi' = \mathrm{Ind}_H^G \theta$.

(cont.) Step 1 $\exists\ \tilde{T} \in \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V)$
s.t. $\tilde{T}|_{1\text{st copy of } W} = T$

$\because$ for $f \in \mathrm{Ind}_H^G W$, define $\tilde{T}f \in V$ by
$\sum_{i=1}^k \pi_{g_i} T f(g_i)$

if f is supported on $g_i H$ then
$f \leftrightarrow g_i \otimes f(g_i)$ in $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$
$\downarrow \tilde{T}$
$g_i T f(g_i)$

$\pi_g \tilde{T} f = \tilde{T} \pi'_g f$: write $g^{-1} g_i = g_{j_i} h_i$

$$\tilde{T}\pi'_g f = \sum_i \pi_{g_i} T f(g^{-1} g_i) = \sum_i \pi_{g_i} T \theta_{h_i^{-1}} f(g_{j_i})$$
$$= \sum_i \pi_{g_i h_i^{-1}} T f(g_{j_i}) = \sum_{g_i h_i^{-1} = g g_{j_i}} \pi_g \pi_{g_{j_i}} T f(g_{j_i}).$$
$$= \pi_g \tilde{T} f.$$

Step 2. any $S \in \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V)$ can be written as $S = \tilde{T}$ for a unique $T \in \mathrm{Hom}_H(W, V)$

$\because$ $T = S|_{1\text{st copy of } W}$.
restr. of $\mathrm{Ind}_H^G \theta$ to the 1st copy of $W = \theta$.
so $T \in \mathrm{Hom}_H(W, V)$
i-th copy of $W = g_i$ (1st copy of W) $\rightsquigarrow$ claim

Rem. this is a particular case of : if $A \xrightarrow{f} B$ is a ring hom, M : A-mod, N : B-mod
$$\mathrm{Hom}_A(M, {}_{f(A)}N) \cong \mathrm{Hom}_B(B \otimes_A M, N)$$
for $A = \mathbb{C}[H]$, $B = \mathbb{C}[G]$.

Rem. $$\chi_{\mathrm{Ind}\,\theta}(g) = \sum_{\substack{i=1,\dots,k \\ g_i^{-1} g g_i \in H}} \chi_\theta(g_i^{-1} g g_i)$$

Prop. If $A < G$ comm. subgrp. (π, V) irred. rep. of G. then $\dim V \leq |G/A|$.

Proof. λ_G, λ_A : reg. reps.

Step 1. $\lambda_G \simeq \mathrm{Ind}_A^G \lambda_A$.

Step 2. $\lambda_G \simeq \bigoplus_{i=1}^{n} \mathrm{Ind}_A^G \varphi^{(i)}$ for $n = |A|$, $\varphi^{(i)} \in \hat{A}$ 1-dim reps of A.

$\because \lambda_A \simeq \bigoplus \varphi^{(i)}$

Step 3 (π, V) is a subrep. of some $\mathrm{Ind}_A^G \varphi^{(i)}$

Step 4 $\dim \mathrm{Ind}_A^G \varphi^{(i)} = |G/A|$.