Exercise 3.2

(ρ, V) irreducible representation of G, $n = \dim V$

$\chi = \chi_\rho$ character of ρ

$g = |G|$

$C = \{ s \in G : \forall t \in G \quad st = ts \}$ center of G, $c = |C|$

(a) $\forall s \in C \ \exists \alpha \in \mathbb{C} : \rho_s = \alpha \, Id_V$, $|\chi(s)| = n$

$st = ts \Rightarrow \rho_s \rho_t = \rho_t \rho_s$ for all t; i.e. ρ_s is an intertwiner

By the irreducibility of ρ & Schur's lemma, ρ_s is a scalar. $\rho_s = \alpha \, Id_V$

$$|\chi(s)| = |Tr \, \rho_s| = |Tr \, \alpha \, Id_V| = |\alpha \, Tr \, Id_V| = |\alpha| \dim V$$

$s^N = e$ for some $N \in \mathbb{N}$ (for example $N = |G|$)

$$\Rightarrow \rho_s^N = Id_V \Rightarrow |\alpha|^N = 1 \Rightarrow |\alpha| = 1$$

(b) $n^2 \leq g/c$

By the irreducibility of ρ and orthogonality of irreducible characters $(\chi, \chi)_{L^2 G} = 1$

i.e. $\frac{1}{g} \sum_{t \in G} |\chi(t)|^2 = 1$

The terms $t \in C$ all contribute by $|\chi(t)|^2 = n^2$ and the rest contribute by non-negative numbers

$\Rightarrow \frac{1}{g} c n^2 \leq \frac{1}{g} \sum_{t \in G} |\chi(t)|^2 = 1$

(c) If ρ is faithful, then C is a cyclic group

By (a), $\forall s \in C \quad \rho_s = \alpha(s) \, \mathrm{Id}_V$ for $\alpha(s) \in \mathbb{C}$, $|\alpha(s)| = 1$

We get a group homomorphism $\alpha : C \to \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$

$s \mapsto \alpha(s)$

(cont.) $\quad p$ is injective $\Rightarrow \alpha$ is also injective

$\Rightarrow G$ is isomorphic to a finite subgroup of $\mathbb{T}$

Any finite subgroup A of $\mathbb{T}$ is cyclic:

if $e^{\frac{2\pi i k}{m}}, e^{\frac{2\pi i l}{n}} \in A$ $\quad k$ & m coprime, l & n coprime

then $e^{\frac{2\pi i}{p}} \in A$ for $p = \operatorname{lcm}(m, n)$

$\because$ first get $e^{\frac{2\pi i}{m}}, e^{\frac{2\pi i}{n}} \in A$ from

$\left(e^{\frac{2\pi i k}{m}}\right)^j = e^{\frac{2\pi i}{m}}$ whenever $jk \equiv 1 \mod m$

(such j exists bc. k & m coprime)

then $\left(e^{\frac{2\pi i}{m}}\right)^a \left(e^{\frac{2\pi i}{n}}\right)^b = e^{\frac{2\pi i}{p}}$ whenever

$am' + bn' = 1$ $\quad m' = p/m, \; n' = p/n$

(possible because m' & n' coprime)

Exercise 5.2

D_n : symmetric group of a regular n-gon

$= \langle r, s : r^n = e,\ s^2 = e,\ srs = r^{-1} \rangle$

$\{ r^k : k = 0, \cdots, n-1 \}$: subgroup of D_n, isomorphic to

C_n (cyclic group of order n)

$D_n = C_n \cup \{ \underbrace{s, sr, \cdots, sr^{n-1}}_{\text{reflections in } D_n} \}$ $\qquad sr^k = r^{-k}s$

χ_h : character (irreducible for $0 < h < \frac{n}{2}$) given by

$$\chi_h(r^k) = 2\cos\frac{2\pi hk}{n}, \quad \chi_h(sr^k) = 0$$

ψ_1, ψ_2 : irreducible (1-dimensional) characters

$$\psi_1(r^k) = 1 = \psi_1(sr^k), \quad \psi_2(r^k) = 1, \quad \psi_2(sr^k) = -1$$

Want to show : $\chi_h \chi_{h'} = \chi_{h+h'} + \chi_{h-h'}$ $\cdots$ (1)

and $\mu_2 = \chi_\alpha$ for $\alpha = (\rho_h \otimes \rho_h)|_{Alt^2 V}$ $\cdots$ (2)

$\chi_{2h} + \mu_1 = \chi_\sigma$ for $\sigma = (\rho_h \otimes \rho_h)|_{Sym^2 V}$ $\cdots$ (3)

$$Alt^2 V = \{ \sum v_i \otimes v_i' \in V \otimes V : \sum v_i' \otimes v_i = - \sum v_i \otimes v_i' \}$$

$$Sym^2 V = \{ \sum v_i \otimes v_i' \in V \otimes V : \sum v_i' \otimes v_i = \sum v_i \otimes v_i' \}$$

(1) $(\chi_h \cdot \chi_{h'})(r^k) = 2\cos\frac{2\pi hk}{n} \, 2\cos\frac{2\pi h'k}{n}$

$= (\omega^{hk} + \omega^{-hk})(\omega^{h'k} + \omega^{-h'k})$ $\quad (\omega = e^{\frac{2\pi i}{n}})$

$= (\omega^{(h+h')k} + \omega^{-(h+h')k}) + (\omega^{(h-h')k} + \omega^{-(h-h')k})$

$= \chi_{h+h'}(r^k) + \chi_{h-h'}(r^k)$

$(\chi_h \cdot \chi_{h'})(sr^k) = 0 = \chi_{h+h'}(sr^k) + \chi_{h-h'}(sr^k)$

(2), (3) $g \in D_n$, λ_1, λ_2 : eigenvalues of $\rho^h(g)$ $(\chi_h = \chi_{\rho^h})$

$\Rightarrow$ $\chi_\alpha(g) = \lambda_1\lambda_2$, $\chi_\sigma(g) = \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2$ (§ 2.1)

$g = r^k$: $\lambda_1 = \omega^{hk}$, $\lambda_2 = \omega^{-hk}$ from $\rho^h(g) = \begin{bmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{bmatrix}$

$g = sr^k$: $\lambda_1 = 1$, $\lambda_2 = -1$ from $\rho^h(g) = \begin{bmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{bmatrix}$

eigenvectors $\begin{bmatrix} 1 \\ \omega^{hk} \end{bmatrix}$, $\begin{bmatrix} 1 \\ -\omega^{hk} \end{bmatrix}$

$\Rightarrow$

(2) : $\chi_\alpha(r^k) = 1 = \psi_2(r^k)$, $\chi_\alpha(sr^k) = -1 = \psi_2(sr^k)$

(3) : $\begin{cases} \chi_\sigma(r^k) = \omega^{2hk} + \omega^{-2hk} + 1 = \chi_{2h}(r^k) + \psi_1(r^k) \\ \chi_\sigma(sr^k) = 1 = \chi_{2h}(sr^k) + \psi_1(sr^k) \end{cases}$