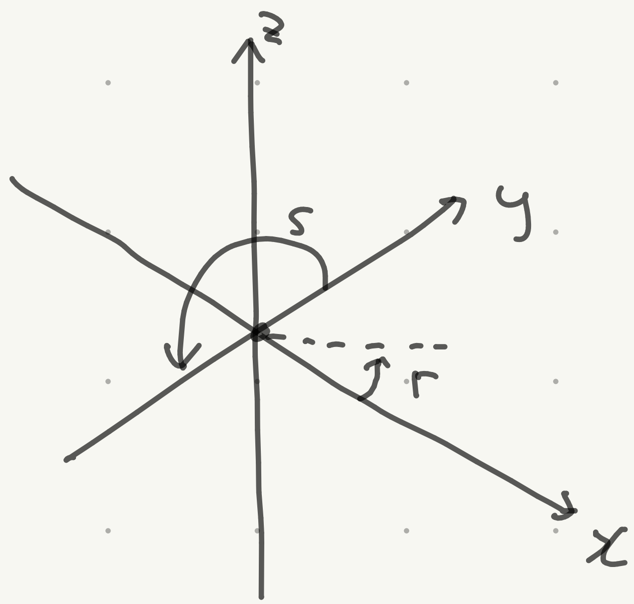
Exercise 5.3

realize D_n as a subgroup of

$SO(3) = \{$ orientation preserving orthogonal linear transforms on $\mathbb{R}^3 \}$ (rotations around origin)

r: rotation around the z-axis, angle $\frac{2\pi}{n}$

s: rotation around the x-axis, angle π



$$X^{(r)} = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X^{(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

invariant subspaces

$W = \{(0,0,u) : u \in \mathbb{R}\} \subset \mathbb{R}^3$, $W' = \{(p,q,0) : p,q \in \mathbb{R}\}$

$X^{(r)}\begin{bmatrix}0\\0\\u\end{bmatrix} = \begin{bmatrix}0\\0\\u\end{bmatrix}$, $X^{(s)}\begin{bmatrix}0\\0\\u\end{bmatrix} = \begin{bmatrix}0\\0\\-u\end{bmatrix}$ $X^{(r)}\begin{bmatrix}p\\q\\0\end{bmatrix} = \begin{bmatrix}p'\\q'\\0\end{bmatrix}$, $X^{(s)}\begin{bmatrix}p\\q\\0\end{bmatrix} = \begin{bmatrix}p\\-q\\0\end{bmatrix}$

$\rightsquigarrow$ restriction on W gives $\rho_r = 1$, $\rho_s = -1$. $(\rho_{r^k} = 1,\ \rho_{sr^k} = -1)$

equal to $\psi(r)$, $\psi(s)$

restriction on W' gives $\rho'_r = \begin{bmatrix}\cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n}\\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n}\end{bmatrix}$

$\rho'_s = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $\rho'_{sr^k} = \begin{pmatrix}\cos\frac{2\pi k}{n} & -\sin\frac{2\pi k}{n}\\ -\sin\frac{2\pi k}{n} & -\cos\frac{2\pi k}{n}\end{pmatrix}$

character $\chi_{\rho'}(r^k) = 2\cos\frac{2\pi k}{n}$, $\chi_{\rho'}(sr^k) = 0$

equal to $\chi_1(r^k)$, $\chi_1(sr^k)$

Rem. if we treat $r \rightsquigarrow X^{(r)}$, $s \rightsquigarrow X^{(s)}$ as complex representation π, we have

$$(\pi, W_{\mathbb{C}}) \simeq \psi_1 , \quad (\pi, W'_{\mathbb{C}}) \simeq \rho^1$$

for $W_{\mathbb{C}} = \{(0,0,u) : u \in \mathbb{C}\}$, $W'_{\mathbb{C}} = \{(p,q,0) : p,q \in \mathbb{C}\}$

$$T : W'_{\mathbb{C}} \to \mathbb{C}^2, \quad (p,q,0) \mapsto (p+iq, p-iq)$$

satisfies $T X^{(g)} = \rho^1_g T \quad g \in D_n$

$g = r$: $T X^{(g)}(p,q,0) = T\left(\cos\frac{2\pi}{n} p - \sin\frac{2\pi}{n} q, \sin\frac{2\pi}{n} p + \cos\frac{2\pi}{n} q, 0\right)$

$$= \left(e^{\frac{2\pi i}{n}}(p+iq), e^{-\frac{2\pi i}{n}}(p-iq)\right) = \rho^1_r T(p,q,0)$$

$g = s$: $T X^{(s)}(p,q,0) = T(p,-q,0) = (p-iq, p+iq)$

$$= \rho^1_s T(p,q,0)$$

Exercise 5.4 (§ 5.7)

$A_4 = \{ \sigma \in S_4 : \text{even permutation} \}$ order $\frac{4!}{2} = 12$

$\cong \{ t, x, y, z : t^3 = e,\ x^2 = e = y^2 = z^2,\ xy = z\ (\Leftrightarrow yz = x, \cdots)$

$t x t^{-1} = y,\ t y t^{-1} = z,\ t z t^{-1} = x \}$

$t \leftarrow (123)$, $x \leftarrow (12)(34)$, $y \leftarrow (13)(24)$, $z \leftarrow (14)(23)$

conjugacy classes of A_4 : $\{e\}$, $\{x, y, z\}$, $\{t, tx, ty, tz\}$, $\{t^2, t^2x, t^2y, t^2z\}$

$tz = yty^{-1}$, $t^2z = xt^2x^{-1}$

$H = \{e, x, y, z\}$ is a normal subgroup, $A_4/H \cong \mathbb{Z}/3\mathbb{Z}$

1-dimensional representations of A_4 from those of $\mathbb{Z}/3\mathbb{Z}$

$\chi_0(\sigma) = 1 \quad \forall \sigma \in A_4$ trivial rep.

$\chi_1(\sigma) = 1,\quad \chi_1(t\sigma) = \omega,\quad \chi_1(t^2\sigma) = \omega^2 \quad (\sigma \in H) \qquad \omega = e^{\frac{2\pi i}{3}}$

$\chi_2(\sigma) = 1,\quad \chi_2(t\sigma) = \omega^2,\quad \chi_2(t^2\sigma) = \omega$

3-dimensional irreducible representation :

$\pi : A_4 \to GL(V)$ $\quad V = \{ \sum_{i=1}^{4} \alpha_i e_i : \sum_i \alpha_i = 0 \} < \mathbb{C}^4$

restriction of permutation representation

$\psi = \chi_\pi$ can be computed as

$$\underbrace{\psi(\sigma) = \#\{\text{fixed points of } \sigma\}}_{\text{char. of perm. rep.}} \underbrace{-1}_{\text{char. of triv. rep. for } \sum_i e_i}$$

$\psi(e) = 3$, $\quad \psi(\sigma) = -1 \quad \sigma = x, y, z$

$\psi(t\sigma) = 0, \psi(t^2\sigma) = 0 \quad \sigma \in H$

Rem. after getting # conjug. classes = 4 and 1-dim. chars χ_0, χ_1, χ_2, we can compute ψ by orthogonality & $\psi(e) \in \mathbb{N}$

We want to show $\pi \simeq \mathrm{Ind}_H^{A_4}\theta$ for the 1-dim rep

$\theta : H \to \mathbb{C}^\times$, $\theta(e) = \theta(x) = 1$, $\theta(y) = \theta(z) = -1$

Strategy 1. put $\psi' = \chi_{\mathrm{Ind}_H^{A_4}\theta}$

set $\psi'(\sigma)$ by $\psi'(\sigma) = \frac{1}{|H|}\sum_{\sigma_i \in A_4} \chi_\theta(\sigma_i\sigma\sigma_i^{-1})$ ↓ same as $\theta(\sigma_i\sigma\sigma_i^{-1})$

$\sigma_i \sigma \sigma_i^{-1} \in H$ ⇐ equiv. to $\sigma \in H$ by H being a normal subgrp

$\rightsquigarrow \psi'(\sigma) = \begin{cases} \theta(\sigma) + \theta(t\sigma t^{-1}) + \theta(t^2\sigma t^{-2}) \\ 0 & \sigma \notin H \end{cases}$

$\sigma = x, y, z \Rightarrow \{\sigma, t\sigma t^{-1}, t^2\sigma t^{-2}\} = \{x, y, z\}$

so $\psi'(e) = 3$, $\psi'(x) = 1 - 1 - 1 = -1 = \psi'(y) = \psi'(z)$

$\psi'(\sigma) = 0$ for $\sigma \in H$ agrees with ψ

Strategy 2 analyze $\mathrm{Ind}_H^{A_4}(\theta, \mathbb{C})$ directly

- underlying sp: $\mathrm{Ind}_H^{A_4}\mathbb{C} = \{ f : A_4 \to \mathbb{C},\ f(\sigma_1\sigma_2) = \theta(\sigma_2) f(\sigma_1)$ for $\sigma_2 \in H \}$

  3-dimensional, determined by $(f(e), f(t), f(t^2))$

- transforms $\pi'_\sigma = (\mathrm{Ind}_H^{A_4}\theta)_\sigma$ $\quad (\pi'_\sigma f)(\sigma') = f(\sigma^{-1}\sigma')$

  $\sigma = x \quad (\sigma^{-1} = x)$ $\quad xt = ty,\ xt^2 = t^2 z$ implies

  $((\pi'_x f)(e), (\pi'_x f)(t), (\pi'_x f)(t^2)) = (f(e), -f(t), -f(t^2))$

  similarly $yt = tz,\ yt^2 = t^2 x$ gives

  $((\pi'_y f)(e), (\pi'_y f)(t), (\pi'_y f)(t^2)) = (-f(e), -f(t), f(t^2))$

  and $zt = tx,\ zt^2 = t^2 y$ give

  $((\pi'_z f)(e), (\pi'_z f)(t), (\pi'_z f)(t^2)) = (-f(e), f(t), -f(t^2))$

(cont.) $(\pi_t' f)(e) = f(t^2)$, $(\pi_t' f)(t) = f(e)$, $(\pi_t' f)(t^2) = f(t)$

from $t^{-1} = t^2$

$((\pi_{t^2}' f)(e), (\pi_{t^2}' f)(t), (\pi_{t^2}' f)(t^2)) = (f(t), f(t^2), f(e))$

Want: intertwiner $\mathrm{Ind}_H^{A_4} \mathbb{C} \to V = \{ \sum_{i=1}^{4} \alpha_i e_i : \sum_i \alpha_i = 0 \}$

look at how H acts:

$f(e)$ is preserved by π_x', flipped by π_y', π_z'

$\rightsquigarrow e_1 + e_2 - e_3 - e_4$

$f(t)$ is preserved by π_z', flipped by π_x', π_y'

$\rightsquigarrow e_1 - e_2 - e_3 + e_4$

$f(t^2)$ is preserved by π_y', flipped by π_x', π_z'

$\rightsquigarrow e_1 - e_2 + e_3 - e_4$

(cont.) this gives a candidate $T : \mathrm{Ind}_H^{A_4} \mathbb{C} \to V$

$$f \mapsto (\varepsilon_1 f(e) + \varepsilon_2 f(t) + \varepsilon_3 f(t^2))e_1 + (\varepsilon_1 f(e) - \varepsilon_2 f(t) - \varepsilon_3 f(t^2)) e_2$$
$$+ (-\varepsilon_1 f(e) - \varepsilon_2 f(t) + \varepsilon_3 f(t^2)) e_3 + (-\varepsilon_1 f(e) + \varepsilon_2 f(t) - \varepsilon_3 f(t^2)) e_4$$

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$

to have compatiblity with π'_t & π_t we should make the coeff. of e_4 invariant under t

$\rightsquigarrow \varepsilon_1 = -1 = \varepsilon_3, \ \varepsilon_2 = 1$

so $$Tf = (-f(e) + f(t) - f(t^2))e_1 + (-f(e) - f(t) + f(t^2)) e_2$$
$$+ (f(e) - f(t) - f(t^2)) e_3 + (f(e) + f(t) + f(t^2)) e_4$$