§2 Lie groups & Lie algebras

Main ref. Fulton–Harris Representation theory

Goal : understand continuous groups.

Model : matrix groups (subgroups of $GL_n(\mathbb{R})$).

Ex. $U(n) = \{ X \in GL_n(\mathbb{C}) : \bar{X}^t = X^{-1} \} < GL_{2n}(\mathbb{R})$

$SU(n) = \{ X \in U(n) : \det X = 1 \}$

$SO(n) = \{ X \in GL_n(\mathbb{R}) : X^t = X^{-1}, \det X = 1 \}$

$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ as

- subgroup of $GL_1(\mathbb{C})$
- $\{ \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} : z \in \mathbb{T} \} \subset SU(2)$, ...

More abstract framework : <u>Lie</u> groups

= manifolds with smooth group structure

manifold : top. sp. s.t. neighborhoods of each point can be modelled by open sets of $\mathbb{R}^n$

smooth grp str : product & inverse maps can be locally represented by C^∞-functions in coordinates.

Ex. $GL_n(\mathbb{R})$: open set of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$.

$\rightsquigarrow$ admits global coord.

$\mathbb{T} = \{ e^{it} : t \in \mathbb{R} \}$ angle coord t is locally well-defined.

Rem. we will stick to matrix groups ; which include

- compact Lie groups $U(n)$, $SO(n)$, ...
- commutative Lie groups $\mathbb{R}^n$, $\mathbb{T} \times \mathbb{R}$, ...
- algebraic groups $GL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, ...

"only" sensible non-example : universal cover of $SL_2(\mathbb{R})$

How to understand cont. grps.

take "infinitesimal model" (Lie algebra)

given by tangent vectors of grp unit $e \in G$.

Example

rotation group $(SO(2))$

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \underset{\text{deriv. at } t=0}{\rightsquigarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

how do we recover the original rotation?

how <u>not</u> to do: integration

$$\int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} dx = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}.$$

how to do: combie with exponenticl

$$\exp\left(\int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} dx\right) = \exp\left(t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= I_2 + t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{t^2}{2}(-I_2) + \cdots$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos t & -\sin t \\ \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \sin t & \cos t \end{bmatrix}$$

Rem. we are computing the fundamental solution of differential eq. $\frac{d v(t)}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v(t)$

(vec. valed)

Quick recap on tangent vectors

M: manifold, $p \in M$

U: neigh. of p, $V \subset \mathbb{R}^n$ open set

$\varphi: V \to U$ homeomorphism.

$\rightsquigarrow$ we can make sense of smooth functions on U:

$f: U \to \mathbb{R}$ s.t. $f \circ \varphi: V \to \mathbb{R}$ is smooth

func $F(x_1, \cdots, x_n)$

<u>tangent vector</u> (directional derivative) at p:

$D: \{\text{smooth funcs around } p\} \to \mathbb{R}$ linear,

$D(f_1 f_2) = D(f_1) f_2(p) + f_1(p) D(f_2)$

"Leibniz rule at p"

Obs.: $D(f)$ is a linear comb. of $\frac{\partial f \circ \varphi}{\partial x_i}$

$\because$ if $p \leftrightarrow 0 = (x_1, \dots, x_n)$ then

$$F(x) = \underset{f(p)}{\overset{\shortparallel}{F(0)}} + \sum \partial_i F(0) x_i + \underbrace{\sum h_i(x) x_i}_{\substack{h_i(0)=0 \\ \text{in } \ker D}}$$

$T_p M = \{ D \text{ as above} \}$

<u>functoriality</u>: if $\varphi: M \to N$ is a smooth map

we get a lin. map $\varphi_\#: T_p M \to T_{\varphi(p)} N$

$(\varphi_\# D)(f) = D(f \circ \varphi)$ for smooth func f around $\varphi(p)$

<u>vector field</u>: distribution of tang. fields, i.e.

$D = (D_p)_{p \in M}$; $D_p \in T_p M$, varies smoothly in p ($D_q(f) = \sum f^i(q) \frac{\partial f}{\partial x_i}$ for some smooth funcs $(f^i)_{i=1}^n$ around p)

Ex. $M = \mathbb{R}$, $D_t f = t f'(t)$

Invariant vector fields on $GL_n(\mathbb{R})$

$X \in M_n(\mathbb{R}) \rightsquigarrow \exp(tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$ $\quad X^0 = I_n$.

write e^{tX}

we have $e^{tX} e^{sX} = e^{(t+s)X}$ etc

for $g \in GL_n(\mathbb{R})$ we get $\tilde{X}_g^{(r)} \in T_g GL_n(\mathbb{R})$ by

$$\tilde{X}_g^{(r)}(f) = \frac{d}{dt} f(g e^{tX}) \Big|_{t=0}$$

Leibniz rule from usual diff. rule for prod. funcs.

Prop. $f_{ij}(g) = g_{ij}$ (i,j)-th component of g

then $\tilde{X}^{(r)}_g(f_{ij}) = \sum_{k=1}^{n} X_{kj} f_{ik}(g)$.

Proof. $\tilde{X}^{(r)}_g(f_{ij}) = \frac{d}{dt} (g e^{tX})_{ij} \Big|_{t=0}$

$= \frac{d}{dt} (g_{ij} + t \sum_k g_{ik} X_{kj} + t^2 \cdots) \Big|_{t=0}$

$= \sum g_{ik} X_{kj} = \sum X_{kj} f_{ik}(g)$ ∎

Consider the left product map $L_h : GL_n\mathbb{R} \to GL_n\mathbb{R}$

$g \mapsto hg$ for some fixed $h \in GL_n\mathbb{R}$

Prop $(L_h)_\# \tilde{X}^{(r)} = \tilde{X}^{(r)}$. $\quad ((L_h)_\# \tilde{X}^{(r)}_g = \tilde{X}^{(r)}_{hg})$

Proof $((L_h)_\# \tilde{X}^{(r)}_g)(f) = \tilde{X}^{(r)}_g(f \circ L_h)$

$= \frac{d}{dt} f(hg e^{tX}) \Big|_{t=0} = \tilde{X}^{(r)}_{hg}$.